9/21/2023

Finish Numerical Integration – Error formulas
Improper Integrals
Review for the Exam on Tuesday

Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)]$$

Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+4f(x_{n-1})+f(x_n)]$$

n must be even for Simpson's rule

Error formulas

$$E_T = \frac{f''(z)(b-a)^3}{12n^2} \le \frac{\max\limits_{on\ [a,b]} |f''(x)|\,(b-a)^3}{12n^2}$$

$$E_S = \frac{f^{(4)}(z)(b-a)^5}{180n^4} \le \frac{\max\limits_{on\ [a,b]} |f^{IV}(x)|\,(b-a)^5}{180n^4}$$

z is some value in the interval [a,b]

Recall:

$$\int_1^2 \ln x\, dx\, , n=4, Trapezoidal\ rule$$

$$\approx \frac{1}{8}[\ln(1)+2\ln(1.25)+2\ln(1.5)+2\ln(1.75)+\ln 2] \approx 0.383699\ldots$$

$$E_T \le \frac{\left|\max\limits_{on\ [a,b]} f''(x)\right|(b-a)^3}{12n^2}$$

$$f'(x)=\frac{1}{x}, f''(x) = -\frac{1}{x^2}$$
$$f''(1) \ge f''(2)$$

$$E_T \le \frac{(1)(2-1)^3}{12(4)^2} = \frac{1}{192} \approx 0.0052\ldots$$

The true error is 0.002594…<=0.0052…

Suppose we want to calculate the integral $\int_1^2 \ln x\, dx$ to within an error of $E \leq 10^{-5}$, how many n do we need to calculate with either Trapezoidal rule or Simpson's rule?

$$10^{-5} \approx \frac{(1)(2-1)^3}{12n^2}$$

$$n^2 \geq \frac{1}{12}10^5 \approx 8333.3333$$

$$n \geq 91.287 \ldots$$
$$n = 92$$

Round up to the next whole number (think of the number 91.2… as a threshold, a minimum…)

With Simpson's Rule

$$E_S \approx \frac{\max\limits_{on\ [a,b]} |f^{IV}(x)|\,(b-a)^5}{180n^4}$$

$$f''(x) = -\frac{1}{x^2}, f'''(x) = \frac{2}{x^3}, f^{IV}(x) = -\frac{6}{x^4}$$

$\frac{1}{x^4}$ is a decreasing function on [1,2], so the maximum is at x=1,

$$10^{-5} \approx \frac{\left(\frac{6}{1^4}\right)(2-1)^5}{180n^4}$$

$$n^4 \geq \frac{6}{180}10^5 \approx 3333.333 \ldots$$

$$n \geq 7.598 \ldots$$

This is a threshold value, must round UP to the next EVEN integer.

$$n = 8$$

Improper Integrals

Integrals of functions where :

1) One of the limits is infinity
2) One of the limits is a point of discontinuity
3) Inside the interval, the function is not continuous

$$\int_0^\infty xe^{-x}dx$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

$$\int_{-1}^1 \frac{1}{\sqrt[3]{x}}dx$$

We deal with using limits. The problem points will be replaced with a variable, and then we'll do the integration normally… and then take the limit as the dummy variable goes to the original value.
If both limits of the integral are problematic, then generally split the integral into two pieces and evaluate the problem limits separately. If the point of discontinuity is in the middle of the interval, we split the integral at that point.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

$$\lim_{a\to 1}\int_0^a \frac{1}{\sqrt{1-x^2}}dx = \lim_{a\to 1} \arcsin x\Big|_0^a = \lim_{a\to 1} \arcsin a - \arcsin 0 = \arcsin 1 = \frac{\pi}{2}$$

When we get a finite value for the integral, we say the integral converges. If we get infinity, then we say the integral diverges.

$$\int_{-1}^1 \frac{1}{\sqrt[3]{x}}dx = \int_{-1}^0 \frac{1}{\sqrt[3]{x}}dx + \int_0^1 \frac{1}{\sqrt[3]{x}}dx$$

$$\lim_{b\to 0}\int_{-1}^b x^{-1/3}dx + \lim_{a\to 0}\int_a^1 x^{-1/3}dx$$

$$\lim_{b\to 0}\frac{3}{2}\left[x^{\frac{2}{3}}\right]_{-1}^b + \lim_{a\to 0}\frac{3}{2}\left[x^{\frac{2}{3}}\right]_a^1 = \frac{3}{2}\lim_{b\to 0} b^{\frac{2}{3}} - (-1)^{\frac{2}{3}} + \frac{3}{2}\lim_{a\to 0} 1^{\frac{2}{3}} - a^{\frac{2}{3}} = 0$$

These can cancel because the limits evaluate to finite numbers (and not infinity). So this integral also converges.

$$\int_0^\infty xe^{-x}dx$$

$$\lim_{b\to\infty}\int_0^b xe^{-x}dx$$

We need integration by parts.

$$u = x, dv = e^{-x}dx$$
$$du = dx, v = -e^{-x}$$

$$\lim_{b\to\infty}\left\{[-xe^{-x}]_0^b - \int_0^b -e^{-x}dx\right\} = \lim_{b\to\infty}\{[-xe^{-x} - e^{-x}]_0^b\} =$$

$$\lim_{b\to\infty}\{-be^{-b} - e^{-b} - (-0e^{-0} - e^{-0})\} = \lim_{b\to\infty}\{-be^{-b} - e^{-b} + 1\} =$$

$$\lim_{b\to\infty}\{-be^{-b}\} + \lim_{b\to\infty}\{-e^{-b} + 1\}$$

$$\lim_{b\to\infty}\{-e^{-b} + 1\} = 1$$

$$\lim_{b\to\infty}\{-be^{-b}\} = -\lim_{b\to\infty}\left\{\frac{b}{e^b}\right\} = -\lim_{b\to\infty}\left\{\frac{1}{e^b}\right\} = 0$$

The integral converges.
If the power of the exponential was positive, the infinities could not cancel, and the integral would have diverged.

Exam #1