9/28/2023

Go over Exam #1
Sequences

A sequence is an ordered list of elements, usually derived from a formula or following a pattern.

$$a_n = \{a_0, a_1, a_2, \dots\}$$

$$\{a_n\}$$

$$a_n = \frac{n^2}{n^3+1}$$

Is an example of an explicit formula for the sequence. $a_n = f(n)$

$$= \{0, \frac{1}{2}, \frac{4}{9}, \frac{9}{28}, \dots\}$$

Another method of writing a sequence is a recursive formula.

$$a_{n+1} = a_n + a_{n-1}$$
$$a_0 = 1, a_1 = 1$$

(these initial terms are referred to as seeds)

$$F_n = \{1,1,2,3,5,8,13, \dots\}$$

Arithmetic sequence
Defined by a common difference between terms

$$\{3, 5, 7, 9, 11, \dots\}$$
$$\{8, 13, 18, 23, 28, \dots\}$$
$$d = a_n - a_{n-1}$$

It has to be the same all the time.

explicit formula=

$$a_n = a_0 + dn$$

Recursive formula=

$$a_{n+1} = a_n + d$$

Geometric sequence
Defined by a common ratio r

$$\{3,6,12,24,48, \dots\}$$
$$\left\{48, -24, 12, -6, 3, -\frac{3}{2}, \dots\right\}$$

Explicit formula=

$$a_n = a_0 r^n$$

Recursive formula =

$$a_{n+1} = ra_n$$

Factorial:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

$$0! = 1$$

$$0! = 1, 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720, 7! = 5040 \ldots$$

$$n!: a_n = n(a_{n-1})$$

Example. Write a formula for the sequence:

$$\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots\right\}$$

$$a_n = \frac{(-1)^n n}{n+1}$$

Example. Write a formula for the sequence:

$$\left\{\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots\right\}$$

$$a_n = \frac{3^n}{3n+1}$$

$\ln(\ln(n))$ has to start at 2 or 3…
$\frac{1}{n-k}$ start above k

What happens to the sequence in the long term? When n gets big?

$$\lim_{n\to\infty} a_n$$

Treat this like a function.

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$$

$$\lim_{n\to\infty} \frac{3^n}{3n+1} = \infty$$

$$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = DNE$$

You can also apply the squeeze theorem just as we did with function limits back in Calc I.

One potential caveat is that $\cos(n\pi) = (-1)^n = \sin\left(n\pi + \frac{\pi}{2}\right) = \sin\left(\frac{2n+1}{2}\right)\pi$

Bounded and monotonic sequences:
Bounded = there is some value for which the sequence never exceeds, and some values for which it never goes below. (both an upper bound, and a lower bound). $min \leq a_n \leq max$

Monotonic = it's always increasing or always decreasing (always non-decreasing or always non-increasing)
Not changing direction
Check the derivative of the explicit formula to see if there is a critical point (n>0)

The theorem says that if a sequence is both bounded and monotonic it must converge.

If a sequence is bounded, but not strictly monotonic, as long as it is monotonic beyond a particular finite value, then the tail of the sequence will have this theorem apply to it.

So this can apply to things like ne^{-n} that might have one critical greater than n=0, but won't apply to $\sin(n)$ (apply the squeeze theorem in this second case)

If the limit of the sequence exists, we say the sequence converges. If it does not exist, then we say it diverges (or if the limit is infinity)

Example.

$$\left\{\frac{4^n}{n!}\right\}$$

Does the sequence converge or diverge?

$$\left\{\frac{1}{1}, \frac{4}{1}, \frac{16}{2}, \frac{64}{6}, \frac{256}{24}, \frac{1024}{120}, \frac{4096}{720}, \frac{16384}{5040}, \frac{65{,}536}{40{,}320}, \frac{262{,}144}{362{,}880}, \dots.\right\}$$

Bounded above by $\frac{64}{6} = \frac{256}{24}$, bounded below by 0
Is it monotonic? Yes, after n=3
Yes, there is a limit (the limit turns out to be 0)

Advice for writing formulas for sequences:
Look for known patterns: 1,2,3,4,5… or 1, 4, 9, 16, etc., or 1, 8, 27, 64, … or 1, 1, 2, 6, 24, 120, etc.
Look for exponentials (geometric sequences), look for arithmetic sequences
Find pattern in the numerator or the denominator separately
Being one off from one of the above patterns… 2, 9, 28, 65, … $n^3 + 1$
If you have combinations of fractions and whole numbers, look for equivalent expressions that help complete the pattern.

$$\frac{2^n}{n} = \{2, 2, \frac{8}{3}, 4, \frac{32}{5}, \dots\}$$

Look for patterns that skip elements of your regular patterns… normally 1, 4, 9, 16, 25, 36, … but maybe you have 1, 9, 25, 49, …

Next time: series, which are sums of sequences