10/1/2024

Analyzing Polynomial Functions
Complex Zeros (Review of Complex Numbers)

Fundamental Theorem of algebra: any n-degree polynomial has n real or complex roots (if you count repetitions), i.e. n total zeros.

Our goal is going to be to identify all the zeros of a polynomial, real or complex, starting with the real ones (starting with rational ones), and then using division to reduce the polynomial to factored form.

All polynomials can be factored in n linear factors, if we allow complex roots. Or, they can be factored into linear factors for the real roots and quadratic factors for the complex roots.

We want the function in factored form to aid in graphing.

Review Complex Numbers

Recall, $i = \sqrt{-1}, i^2 = -1$

Compare to powers of (-1)… $(-1)^1 = (-1)^3 = (-1)^5 = -1, (-1)^2 = (-1)^4 = (-1)^6 = 1$

For i there is a 4-fold cycle:

$$i = i^5 = i^9 \ldots.$$
$$i^2 = -1 = i^6 = (i^2)(i^4) = (-1)((-1)^2) = -1 = i^{10} \ldots$$
$$i^3 = i(i^2) = -i = i^7 = i^{11} \ldots$$
$$i^4 = (i^2)^2 = (-1)^1 = 1 = i^8 = i^{12} = \cdots$$

$$i, -1, -i, 1, i, -1, -i, 1, \ldots$$

Imaginary numbers: contain multiples of i
Complex numbers: real part and an imaginary part
Standard form:

$$z = a + bi$$

$$Re(z) = a$$
$$Im(z) = b$$

Complex numbers apply normal order of operation, but standard form does not contain any i^2

$$(3 - 2i) - (4 - 3i) = 3 - 2i - 4 + 3i = -1 + i$$

$$(3 - 2i)(4 - 3i) = 12 - 9i - 8i + 6i^2 = 12 - 9i - 8i - 6(-1) = 18 - 17i$$

$$\bar{z} = a - bi$$

$$\frac{2 - i}{3 + i} \times \frac{3 - i}{3 - i} = \frac{6 - 2i - 3i + i^2}{9 - 3i + 3i - i^2} = \frac{5 - 5i}{10} = \frac{1}{2} - \frac{1}{2}i$$

Magnitude of a complex number:

$$r = \|z\| = \sqrt{a^2 + b^2}$$

$$\sqrt{-4}\sqrt{-50} = \sqrt{(-1)(4)}\sqrt{(-1)(50)} = \sqrt{-1}\sqrt{4}\sqrt{-1}\sqrt{50} = (i2)\left(i\sqrt{25 \times 2}\right) = 2i\left(5i\sqrt{2}\right) = 10i^2\sqrt{2}$$
$$= -10\sqrt{2}$$

Polynomials with real coefficients must have complex zeros that come in conjugate pairs.

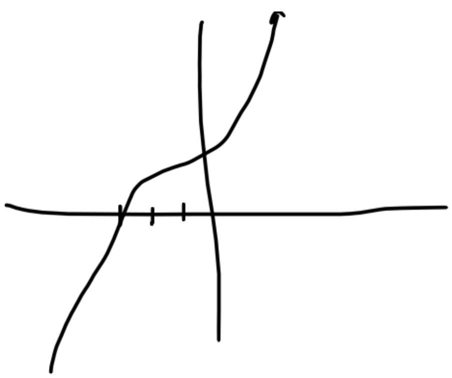
Write the polynomial in factored form. List all real and complex zeros. Sketch the graph.

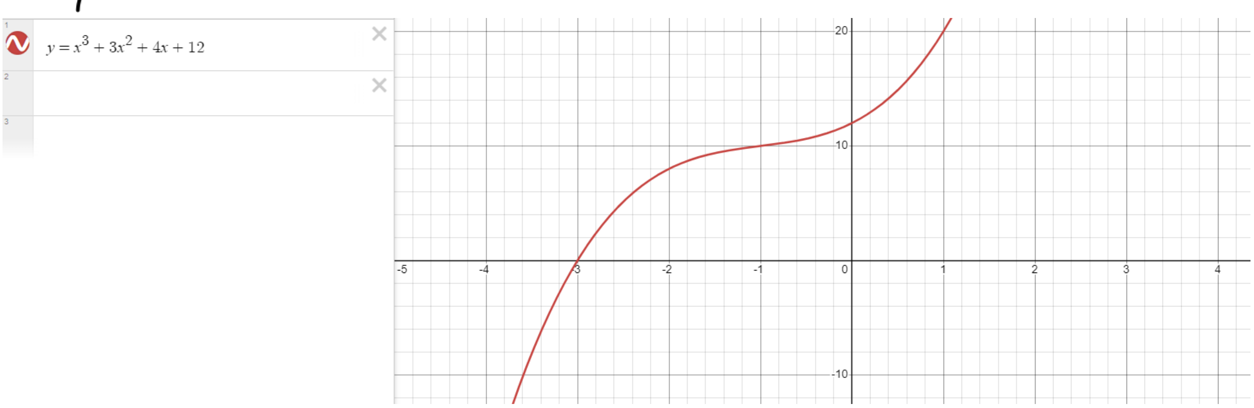
$$p(x) = x^3 + 3x^2 + 4x + 12$$

$$x^2(x+3) + 4(x+3) = (x^2+4)(x+3)$$

$$x + 3 = 0, x = -3$$
$$x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm\sqrt{-4} = \pm 2i$$

Three zeros: -3, 2i, -2i





$$p(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$$

Constant: 3; factors are 1, 3

Leading coefficient: 4; factors are 1, 2, 4

$$rational\ zeros: \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$$

Rule of signs: postive: 4 or 2 or 0 positive real roots

$$4x^4 + 4x^3 + 13x^2 + 12x + 3$$

No sign changes, no negative real roots

$(x-1)$

$$\begin{array}{r|rrrrr} 1 & 4 & -4 & 13 & -12 & 3 \\ & & 4 & 0 & 13 & 1 \\ \hline & 4 & 0 & 13 & 1 & \boxed{4} \end{array} \quad X$$

$(x-3)$

$$\begin{array}{r|rrrrr} 3 & 4 & -4 & 13 & -12 & 3 \\ & & 12 & 24 & 111 & 297 \\ \hline & 4 & 8 & 37 & 99 & \boxed{300} \end{array} \quad X$$

$(x-\frac{1}{2})$

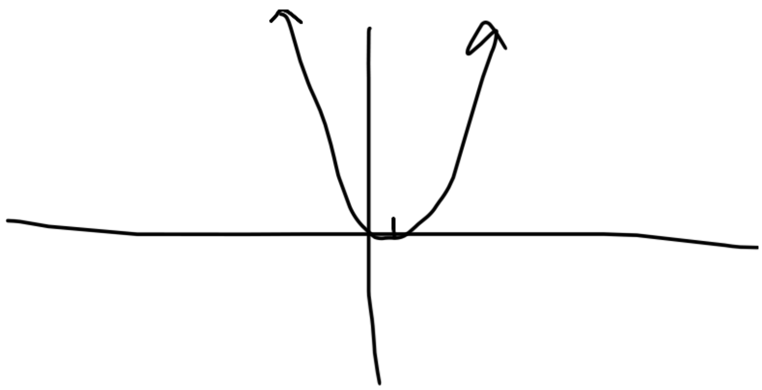
$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & 13 & -12 & 3 \\ & & 2 & -1 & 6 & -3 \\ \hline & 4 & -2 & 12 & -6 & \boxed{0} \end{array} \quad \checkmark$$

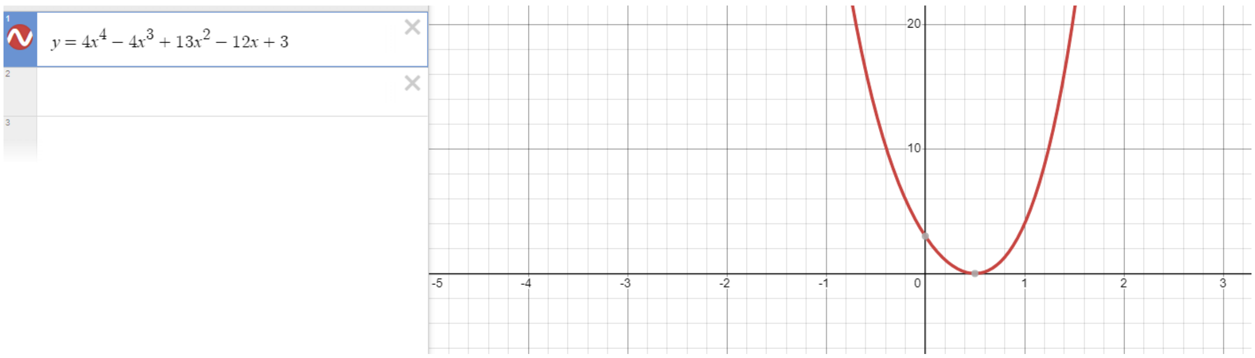
$$\left(x-\frac{1}{2}\right)(4x^3 - 2x^2 + 12x - 6) = 2\left(x-\frac{1}{2}\right)(2x^3 - x^2 + 6x - 3) = (2x-1)(2x^3 - x^2 + 6x - 3)$$

$$(2x-1)[x^2(2x-1) + 3(2x-1)] = (2x-1)(2x-1)(x^2+3) = (2x-1)^2(x^2+3)$$

Zeros:

$$2x - 1 = 0, x = \frac{1}{2}\ (multiplicity\ 2)$$
$$x^2 + 3 = 0, x = \pm i\sqrt{3}$$





Suppose that we want to construct a polynomial with a given set of zeros, a given degree of the polynomial, and a given value of the function.

Suppose you want to construct a polynomial of degree 5 with the roots 2 (multiplicity 2), -1 and 3-2i. And satisfied the condition that $p(0) = 45$

$$(x-2)^2(x+1)\big(x-(3-2i)\big)\big(x-(3+2i)\big) = (x-2)^2(x+1)(x-3+2i)(x-3-2i)$$

$$(x-3-2i)(x-3+2i) = x^2 - 3x + 2ix - 3x + 9 - 6i - 2ix + 6i - 4i^2 = x^2 - 6x + 13$$

$$p(x) = a(x-2)^2(x+1)(x^2-6x+13)$$

$$45 = a(-2)^2(1)(13) = a52$$

$$a = \frac{45}{52}$$

$$p(x) = \frac{45}{52}(x-2)^2(x+1)(x^2-6x+13)$$