10/3/2024

Rational Functions

Definition of a rational function:

$$f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$$

Polynomial have a domain of all real numbers.
Since we can't divide by zero, the domain of the rational function will be impacted by any zeros of the denominator $q(x)$.

$$r(x) = \frac{x^2 + 5x - 6}{x^2 + 7x + 12}$$

The domain of this rational function will depend on the denominator not being equal to zero.

$$x^2 + 7x + 12 = 0$$

$$(x + 3)(x + 4) = 0$$

The denominator will be equal to 0 when x= -3, or -4.

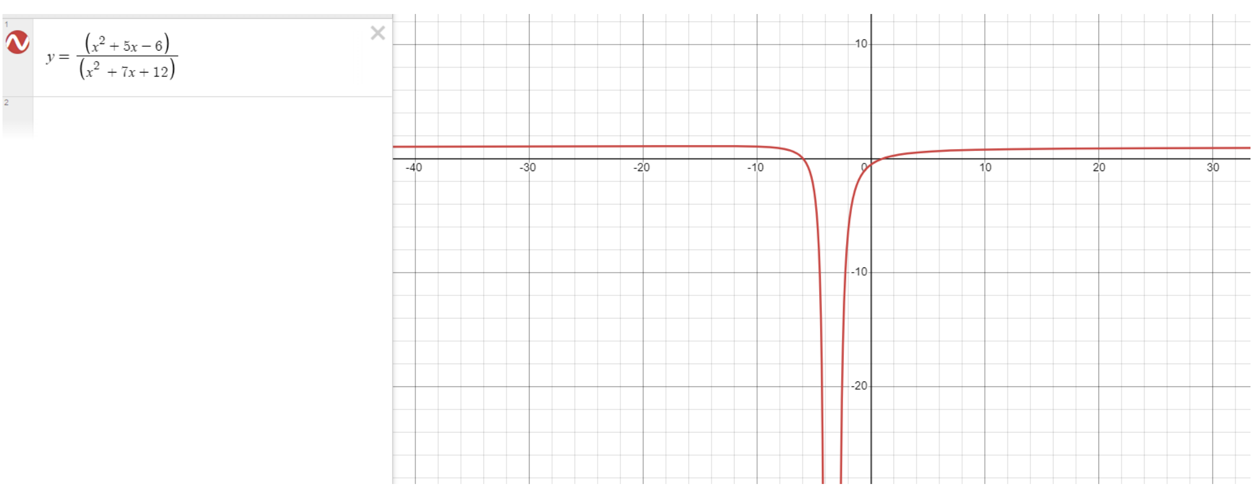
The domain of the rational function must exclude these values:

$$x \neq -3, -4$$

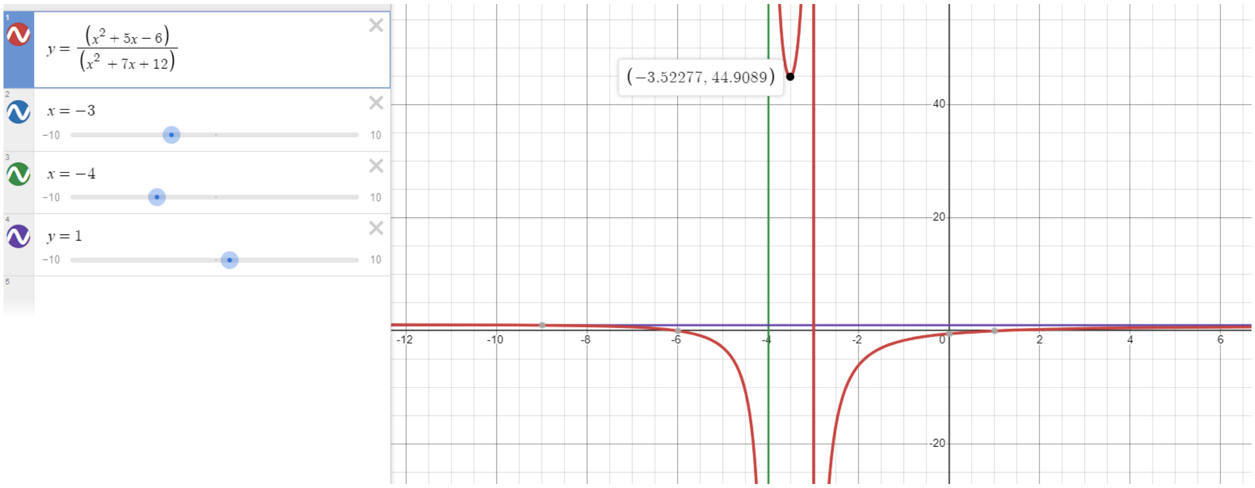
$$(-\infty, -4) \cup (-4, -3) \cup (-3, \infty)$$

To find the range: for some simple functions (not the one above), the range can be determined by finding the inverse function, and then domain the inverse function is the range of the original function.

Given the squares in this function, that suggests that this function is probably not one-to-one, so finding the inverse will not be easy (or doable if at all). In such a circumstance, graphing it is the best option.



$$r(x) = \frac{(x+6)(x-1)}{(x+3)(x+4)}$$



The range, it's not all real numbers.
Some graphing programs will plot the "vertical asymptote"

The approximate range:

$$(-\infty, 1) \cup [44.9089, \infty)$$

Rational functions can have asymptotes:
Lines (or other functions) that a graph approaches but does not cross (after some value of x)

Vertical Asymptotes:
A vertical asymptotes will occur in a rational function when the polynomial in the denominator is equal to zero, and the factor that creates that zero does not cancel with any factor in the numerator.

Hole: is created from a rational function that has a factor in the numerator and a factor in the denominator that exactly cancel. The function is still undefined at the at point (because the denominator is zero before simplification), but the rest of the graph will act as though it is like the reduce rational function, but it will skip over the hole.

The vertical asymptote is a vertical line of the form x=b, where the graph approaches but cannot cross. Typically, the graph goes to infinity or negative infinity as it gets closer to the asymptote.

As you get closer to the asymptote, the value in the denominator goes to 0… dividing by a small number, this is like multiplying by a large number (its reciprocal).

Horizontal Asymptotes:
Are horizontal lines of the form $y = a$ that the graph will approach for large values of x.

If the polynomial in the numerator is the same degree or smaller than the polynomial in the denominator, then there will be a horizontal asymptote.

a) When the degree of the numerator is smaller than the degree of the denominator, then the horizonal asymptote is $y = 0$

b) When the degree of the numerator is the same as the degree of the denominator, the horizontal asymptote is determined by dividing the leading term of the numerator by the leading term of the denominator.

$$r(x) = \frac{x^2 + 5x - 6}{x^2 + 7x + 12}$$

$$y = \frac{x^2}{x^2} = 1$$

Slant or Oblique Asymptote:
This occurs when the numerator is exactly one degree higher than the degree of the denominator. Is a straight line and will be of the form $y = mx + b$

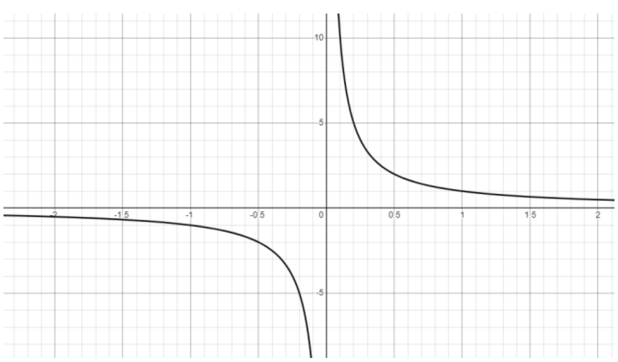
To find this asymptote, you have to do long division (synthetic division sometimes).

You can have asymptotes that are higher degree polynomials, however, we don't cover that in this book.

Not all rational functions have holes in the domain. You can have polynomial denominators that have no zeros (sum of squares, has only complex zeros).

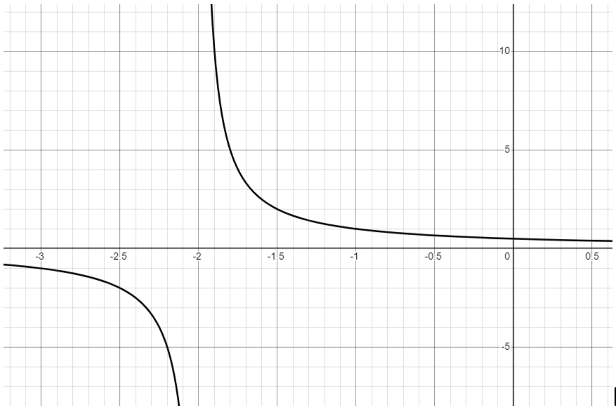
Start with the simplest possible case.

$$f(x) = \frac{1}{x}$$



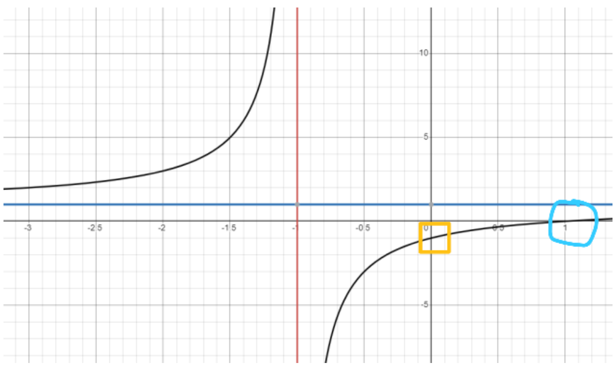
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Vertical asymptote at x=0
Horizontal asymptote at y=0

$$f(x) = \frac{1}{x + 2}$$



Domain: $(-\infty,-2) \cup (-2,\infty), VA: x = -2$

$$f(x) = \frac{x-1}{x+1}$$



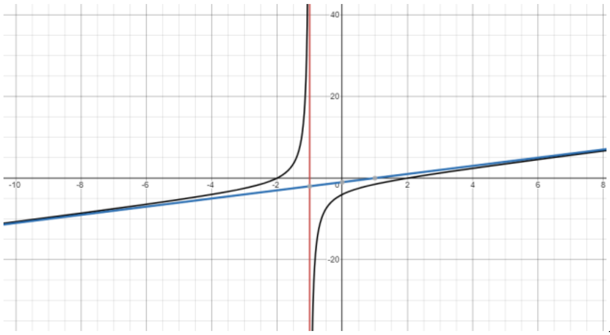
Domain: $(-\infty,-1) \cup (-1,\infty)$, Range: $(-\infty,1) \cup (1,\infty)$
VA: $x = -1$, HA : $y = 1$
Intercepts : x-intercept is when y=0, or when the numerator is equal to 0., x=1
y-intercept is when x=0, in this case it's (0,-1)

$$f(x) = \frac{x^2-4}{x+1}$$

-1	1	0	-4
		-1	1
	1	-1	-3

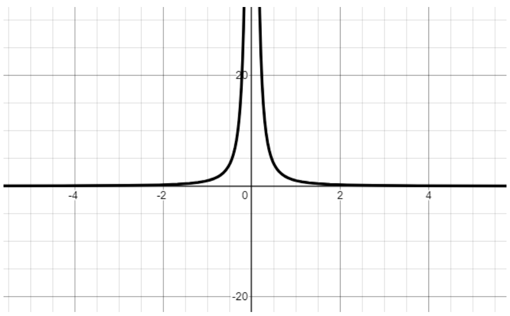
$$f(x) = x - 1 - \frac{3}{x+1}$$

VA: $x = -1$
Slant asymptote: $y = x - 1$



$Domain: (-\infty - 1) \cup (-1, \infty), Range: all\ reals$

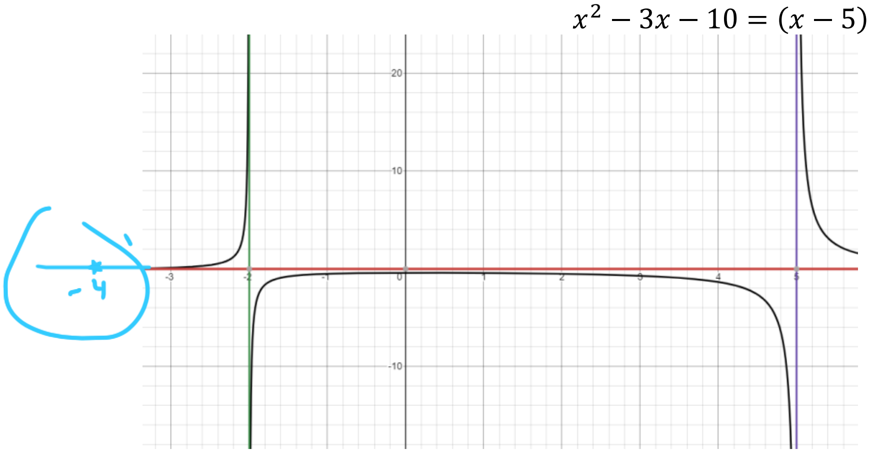
$$f(x) = \frac{1}{x^2}$$



Range: $(0, \infty)$

$$f(x) = \frac{x + 4}{x^2 - 3x - 10}$$

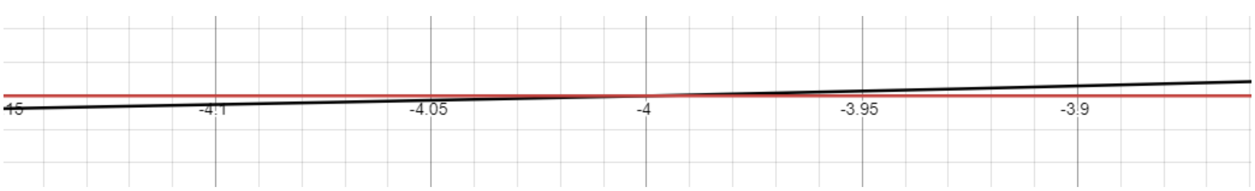
$$x^2 - 3x - 10 = (x - 5)(x + 2)$$



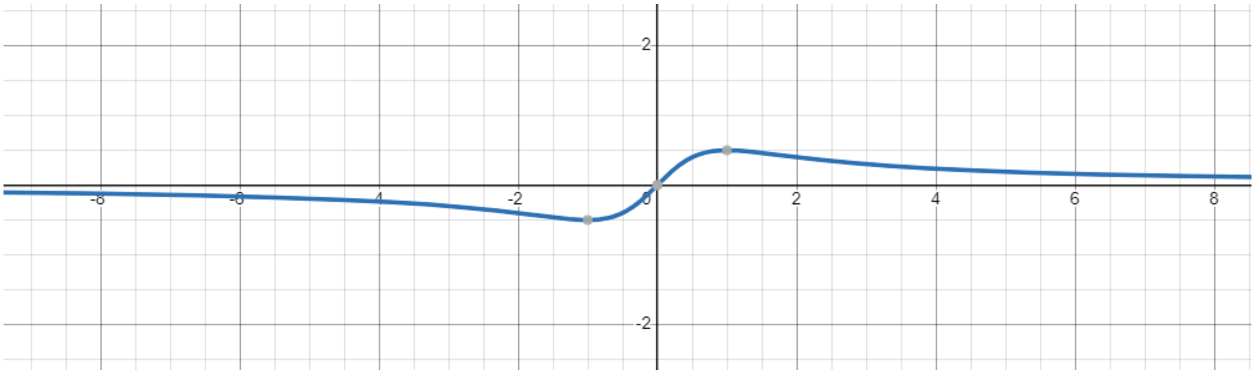
VA: x=5, x=-2, HA: y=0

Domain: $(-\infty, -2) \cup (-2,5) \cup (5, \infty)$, Range: all reals

X-intercept at x= -4 off the graph.

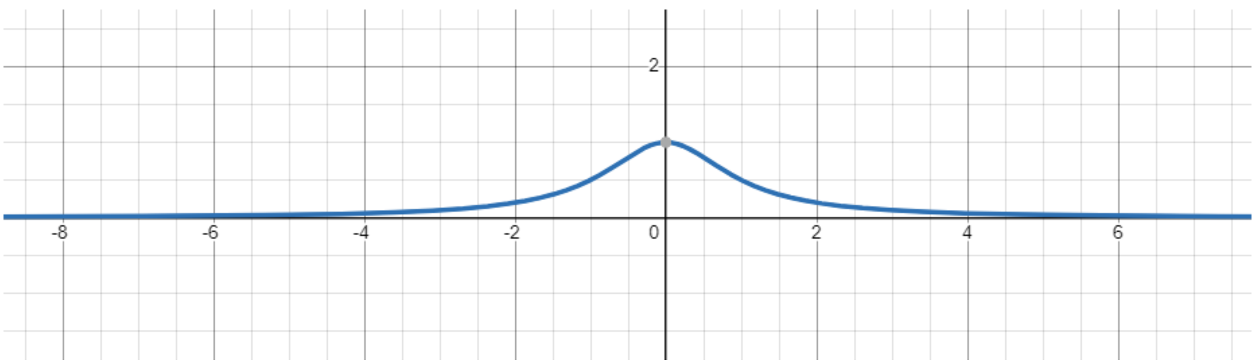


$$f(x) = \frac{x}{x^2 + 1}$$



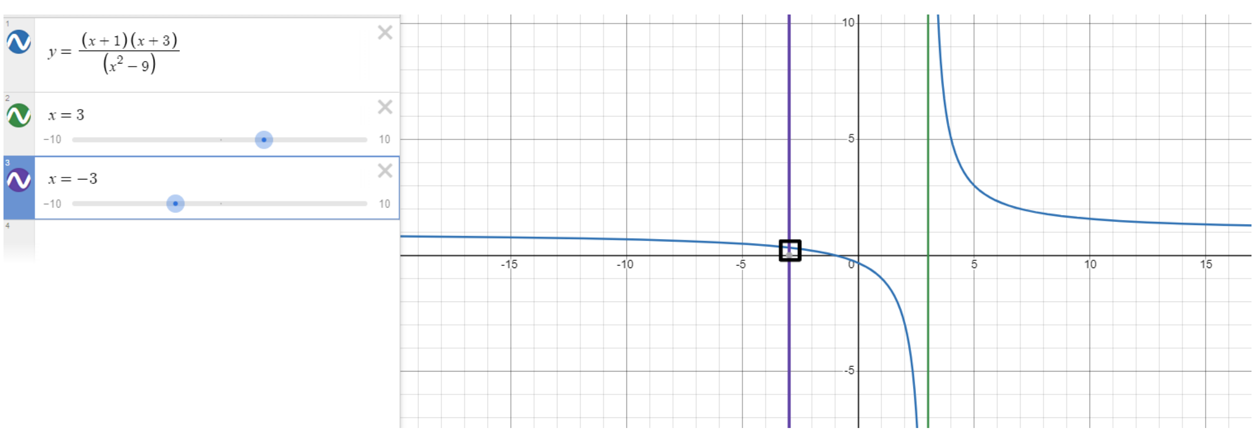
Domain: all real
Range: [-1/2,1/2]
Intercept: (0,0)

$$f(x) = \frac{1}{x^2 + 1}$$



No VA, only HA: y=0

Next time: use the things we talked about today to graph functions by hand without technology.



This graph has a hole at x= -3.