Factoring

Factoring is one of the most important skills we will learn in beginning algebra. It's a skill needed to solve all polynomial equations higher than linear, and to solve or simplify both rational and radical problems. It never goes away. It's usually best to start out with terms in some kind of order: either ascending order (starting with the constant and working up to higher powers in order), or descending order (starting with the highest power and working down to the constant). It's typical to use descending order, and I will for all these problems.

Step 1. The most fundamental step in factoring problems is to start with finding any common factors. Finding a greatest common factor doesn't mean you're done. Further factoring may be required, but we start here to make the rest of the factoring simpler.

Example 1. $x^4y - xy^4$
When dealing with variables like this, the easiest way to spot a greatest common factor (GCF) is the find the lowest degree of the variable over all the terms. Here, the lowest degree of y (given that y's are in every term) is 1, and the lowest degree of x (given that x is in every term) is also 1. Thus the GCF is xy. To factor this out, divide each term by the GCF. Put the result of that division in a parentheses multiplied by the GCF.

$$\frac{x^4y - xy^4}{xy} = \frac{x^4y}{xy} - \frac{xy^4}{xy} = x^3 - y^3$$

Thus $x^4y - xy^4$ factors as $xy(x^3 - y^3)$.

We will finish factoring this problem in Example 3.

Example 2. $3x^2 + 9x - 54$
This polynomial could be factored with the 3 left alone, but it'll be a lot easier to work with if we take all the unnecessary constants out of the problem.

First, look at the variables. Every term doesn't have a variable, so we know we can't factor out any x's. (This is a common error, so watch out for it.) Then look at the constants. Start with the factors of the smallest constant/coefficient and factor that. Here, it's just 3, which is prime. Check if this number, or any of its factors, divides into ***all*** the other constants. As before, divide all the terms by the number you pick and make sure it divides evenly.

$$\frac{3x^2 + 9x - 54}{3} = \frac{3x^2}{3} + \frac{9x}{3} - \frac{54}{3} = x^2 + 3x - 18$$

As before, the 3 doesn't disappear. It goes outside the parentheses with the result of the above division going into the parentheses: $3(x^2 + 3x - 18)$.

We'll finish the rest of this problem in Example 10.

Sometimes all you can factor out if the GCF, but outside the initial introduction to this technique, you should expect to check the factor in parentheses to see if it can be factored further. Don't assume you are done once you get the GCF.

Step 2. Check the problem next for special formulas. They will save you time trying to do the problem by hand when you don't need to. There are 3 types of special formulas to consider: sum and difference of cubes, difference of squares, and perfect square trinomials. I'll list them first, and then we'll do examples of each one.

Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ (1)
Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ (2)

Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$ (3)

Perfect Square Trinomial: $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$ (4)

The formulas that come in pairs differ only by + and – signs that I've highlighted. It can reduce the memorizing if you remember just one of the pairs and know how the signs change.

Example 3. $x^3 - y^3$
This is the rest of the problem from Example 1.

This is a difference of cubes. Any powers of variables that are evenly divisible by 3 qualify as a cube such as x^3, x^6, x^9, etc. Both terms are cubes so we use formula (2) from above. a=x and b=y. So our problem factors as:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Combining this with the result from Example 1, we have that $x^4y - xy^4$ factors completely as $xy(x - y)(x^2 + xy + y^2)$.

One other thing that should be mentioned is that the result of a sum/difference of cubes factoring, the trinomial that results is never factorable; it's always prime.

Example 4. $16q^3 + 54p^3$

When you have 2 terms, you are either going to have a sum/difference of cubes or sum/difference of squares. (Sums of squares are not factorable, but all the other ones are.) Before we can proceed, though, we need to first check for any GCFs. The variables are different in both terms, so there is no common variable to remove. Both terms are even, though, so at least a 2 can come out. 54 only divides by 2 once, so we can only remove the one 2. That leaves us with $2(8q^3 + 27p^3)$.

Now check the parentheses for things to be factored further. Our variables are cubes, but to be factored, the coefficients/constants will also have to be perfect cubes. It's helpful to know at least the first 5 or 6 perfect cubes: $1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, 5^3 = 125, 6^3 = 216, 7^3 = 343$. It's very uncommon to see 216 and 343, but the others are pretty common for these kinds of problems. For the coefficients, now that the common factors have been removed, we want to check to see if both of them are on this list. They are. That means $8q^3 + 27p^3 = (2q)^3 + (3p)^3$, so to apply our formula (1), a =2q and b=3p. This gives us:

$$8q^3 + 27p^3 = (2q + 3p)[(2q)^2 - 2q * 3p + (3p)^2] = (2q + 3p)(4q^2 - 6pq + 9p^2)$$

So in the end, we have the complete factoring:

$$16q^3 + 54p^3 = 2(2q + 3p)(4q^2 - 6pq + 9p^2)$$

As with Example 4, the trinomial is not factorable. And the linear factor is as simple as it can be.

Example 5. $x^4 - 1$

First, check that there are no GCFs we can factor out first. There are none, so we check our formulas. Any even powers (ones that are divisible evenly by 2) are perfect squares, so x^2, x^4, x^6, x^8, etc. (When dealing with something like x^6 or x^{12} that can appear on both the cube and square lists, it's generally best to treat the variable as a square first, and cube second, so check for other squares if possible, before checking for cubes. This is generally only an issue in very difficult problems.) Here, x^4 is on our list of perfect squares and not on the cube list, so we treat it like a square. What about the constant? As with the cubes, if we hope to factor this as a difference of squares, we want the constant to be a square. We will see many more of these than we will cubes. I've included the table below of the common ones:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**20**	**25**
n^2	1	8	9	16	25	36	49	64	81	100	121	144	169	196	225	256	400	625

Everything up to 13, along with 15 and 25 are pretty common. Some of the others are less so. If you aren't sure if a large number is a perfect square, you can also check it in your calculator.

In this case, 1 is a perfect square, so we have two squares and a minus sign between them, that means, it can be treated like a difference of squares with $x^4 = (x^2)^2$.

Using formula (3) from above, we have: $x^4 - 1 = (x^2 + 1)(x^2 - 1)$

It's tempting to think we're done, but before we can conclude that, we have to check to see if we can't factor any of the resulting polynomials. As it turns out, $x^2 + 1$ is a sum of squares, which is never factorable: it's always prime. But the other factor, $x^2 - 1$ is a difference of squares still, so we will have to factor that one more time.

$$x^2 - 1 = (x - 1)(x + 1)$$

Replacing into the previous step, that means:

$$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x - 1)(x + 1)$$

Linear factors with no GCFs are our final goal in factoring problems whenever possible.

Example 6. $16m^2 - 72mn + 81n^2$
There are no GCFs in this problem. Only 2s divide into the first coefficient, and only 3s into the last term. But both 16 and 81 are perfect squares from our table in Example 5. That means there is a chance that this is a perfect square trinomial, and that's much easier to factor than other trinomials with a leading coefficient. We need to check to see if the middle term matches the formula in (4). $16m^2 = (4m)^2$ and $81n^2 = (9n)^2$. That means for our formula a=4m and b=9n. What we need to know is if the middle term is equal to 2ab.

$$2ab = 2(4m)(9n) = 72mn$$

This checks out, and since we have a minus sign in the expression, we'll use the second version of (4). Thus:

$$16m^2 - 72mn + 81n^2 = (4m - 9n)^2$$

You can check this by FOILing out (4m-9n)(4m-9n).

Before we do trinomial factoring in general, there is one other special type of factoring to look for: polynomials of 4 terms. For this, we want to try factoring by grouping.

Example 7. $pq + 5q + 2p + 10$

When you have a set of terms like this, we are going to group the term into sets of 2: $(pq + 5q) + (2p + 10)$. We'll take each group separately and look for any GCFs.

pq+5q has a common factor of q: q(p+5)
2p+10 has a common factor of 2: 2(p+5)

What we want, if this procedure will help us, is for the two sets of parentheses in each case to be exactly the same. If it's not, we can't proceed by this method. Here, we have p+5 in both cases.

$$pq + 5q + 2p + 10 = q(p + 5) + 2(p + 5)$$

Since both terms have (p+5) as a factor, we can treat this as the GCF of the new expression and factor that out. The bits that are left sitting outside the pair of parentheses are collected in the new set of parentheses.

$$q(p + 5) + 2(p + 5) = (p + 5)(q + 2)$$

It is sometimes tempting for students to want to put a 2 somewhere because there are two sets of (p+5), but we aren't adding these, we are factoring them. It's like combining like terms, with the part that is "like" being the entire set of parentheses. Compare this problem with a related problem: replace (p+5) with u.

$$q(p + 5) + 2(p + 5) = qu + 2u$$

If I want to factor these two terms, I look for what they have in common. What they have in common in this form is u. So, I factor u out and divide each term by u:

$$qu + 2u = u\left(\frac{qu}{u} + \frac{2u}{u}\right) = u(q + 2)$$

That's not 2u, just u. And if we put u=p+5 back into the equation, we have the (p+5)(q+2) that we found above. Remember to treat the parentheses as a single thing, not as 2 different things.

If in doubt, remember you can check your answer by remultiplying to see if you come back to the original problem. We'll do this a lot when we start working with trinomials.

Example 8. $x^3 + 3x^2 - 5x - 15$
When we don't have a sum/difference of cubes, cubes can be very hard to factor in general. The only other way we'll learn to deal with cubes until college algebra is through factoring by grouping. Group the terms in sets of 2.

$x^3 + 3x^2$ has a common factor of x^2 since this is the lowest power in the two terms: $x^2(x+3)$
-5x – 15 has a common 5. But since the leading term of this pair is negative, let's factor out the negative as well. Do this even if the second term of this pair is positive: -5(x+3)

$$x^3 + 3x^2 - 5x - 15 = x^2(x+3) - 5(x+3)$$

As with our previous example the expression in the parentheses is identical as it must be for this method to work. Factor out the common (x+3) and leave in the second parentheses whatever is left over.

$$x^3 + 3x^2 - 5x - 15 = x^2(x+3) - 5(x+3) = (x+3)(x^2-5)$$

Whenever you see a set of parentheses containing a polynomial that is not linear, you should check it to see if it can be factored further. In this example, the x^2 suggests we might be looking at a difference of squares, but 5 is not a perfect square and it can't be factored any further if we want to keep working with integers.

Step 3. Once you've considered all the special products and other special cases like factoring by grouping, and none of those apply, we are left with just trinomials. Factoring trinomials can be easy or complicated depending on the coefficients, so it's the last thing you consider. We'll start with some easy cases, where the leading term has a (unstated) coefficient of 1.

Example 9. $x^2 + 8x + 12$
When the leading coefficient is one, then the only factors we have to concern ourselves with are the factors of the final constant. Since the final constant is positive, what we are looking for is a pair of factors that multiply to get 12, but which add to become the middle coefficient 8.

What are the pairs of factors of 12 available? 1x12, 2x6, 3x4. That's it. The pair that adds to 8 is 2 and 6. Since the middle term is positive, we'll use positive signs in our factoring:

$$x^2 + 8x + 12 = (x+2)(x+6)$$

FOIL to check that this does work.

Example 9B. $x^2 - 8x + 12$
How does this problem differ from the last one? Not at all except that we want factors of 12 that add to 8, but since 8 is negative, the factors with both have the negative sign.

$$x^2 - 8x + 12 = (x - 2)(x - 6)$$

As long as that last constant is positive, the signs in both factors must match, since 2x6=12, but so does (-2)x(-6)=12. The sign comes from where they add, which is the middle term.

Example 10. $x^2 + 3x - 18$
What about if the last constant is negative? Here we'll still need factors of 18, but the negative sign on the end is an indication that the factors we want each have a different sign, one is positive and one is negative. When the results get added together, this has the effect of the middle term being the difference of the two factors. Factors of 18 are 1x18, 2x9 and 3x6. The pair of factors that has a difference of 3 is 3x6. To get the + in the middle term, the larger of the two factors takes that sign.

$$x^2 + 3x - 18 = (x - 3)(x + 6)$$

You can check this by FOILing: $(x - 3)(x + 6) = x^2 + 6x - 3x - 18 = x^2 + 3x - 18$ just was we wanted.

Thus, to factor Example 2 completely: $3x^2 + 9x - 54 = 3(x - 3)(x + 6)$.

Example 11. $2m^2 + 7m + 3$
When the leading coefficient isn't 1 and it can't be factored out (as it was in Example 2), things get more complicated and there are two methods one can use. The first is called "trial and error", and the second is an extension of factoring by grouping. When the coefficient/constant on either end doesn't have a lot of factors, there isn't a lot of combinations to check, it's not a bad method. And with practice, you can get good at spotting ways to eliminate options you don't have to both checking. But when the factors get really large and there are many options, factoring by grouping may be more efficient for the beginner.

We'll do the first two examples by trial and error, and then the third example with grouping.

Here, 2 only factors as 2x1, and the final constant 3 only factors as 3x1. There are only two ways they can combine in a parentheses:

$$(2m \quad 1)(m \quad 3)$$
$$(2m \quad 3)(m \quad 1)$$

More specifically, they can actually only be:

$$(2m+1)(m+3)$$
$$(2m+3)(m+1)$$

Since all the constants in the polynomial we are factoring are positive. (This follows the same rule of signs we saw in Examples 9 and 10.)

When using trial and error, the only thing to do here is to FOIL and check the result of both forms to see which one produces our original polynomial. And really, since the factors were chosen to satisfy the two end terms (the F and the L of FOIL), we just need to check which combination produces the middle terms (the O and the I of FOIL).

The (2m+1)(m+3) gives OI terms of 6m+m.
The (2m+3)(m+1) gives OI terms of 2m+3m.

The first pair adds to 7m so that's the one we keep as the answer.

$$2m^2+7m+3=(2m+1)(m+3)$$

Example 12. $24x^2-71x-30$
Trial and error is much more problematic in a case like this one because 24 can be factored as 1x24, 2x12, 3x8 and 4x6, and 30 can be factored as 1x30, 2x15, 3x10, and 5x6. To make matters still worse, the last constant is negative, which means we also have to deal with two options for the signs. To find all the cases we need to test we need to be methodical. I always start with the factors that are closest together (4x6 and 5x6) and work my way out to factors that are less similar. Unless one of your coefficients is truly huge, the more similar ones tend to be more frequently represented in textbook problems.

$$(4x-5)(6x+6) \text{ or } (4x+5)(6x-6)$$
$$(4x-6)(6x+5) \text{ or } (4x+6)(6x-5)$$

Notice that what I did was combine the 4x6 and 5x6 needed to achieve 24 and 30 on either end, and put them together with the two possibilities for alternating signs each.

As it turns out, I don't need to check any of these combinations by FOILing because I can eliminate them for other reasons. The polynomial I started with has no common factors, but (4x±6) and (6x±6) do have common factors. If these factors worked, the common 2 or the common 6 would show up even after I did the FOILing. So these can't work.

On to the next pair. Let's try 3x8 together with 5x6. Our combinations then are:

$$(3x-5)(8x+6)\ or\ (3x+5)(8x-6)$$
$$(3x-6)(8x+5)\ or\ (3x+6)(8x-5)$$

These 4 combinations have the same problem the last set did. (3x±6) and (8x±6) have common factors of 3 or 2 respectively. None of these can be the right answer.

Now let's try 3x8 with 3x10

$$(3x-3)(8x+10)\ or\ (3x+3)(8x-10)$$
$$(3x-10)(8x+3)\ or\ (3x+10)(8x-3)$$

The top pair has the same common factor problem we saw before, but the bottom pair doesn't, so let's FOIL these out and see what we get.

(3x-10)(8x+3): the OI terms are 9x-80x = -71x
(3x+10)(8x-3): the OI terms are -9x+80x = +71x

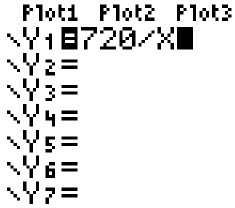
The (3x-10)(8x+3) gives us the middle term we want. Now that we've found it we can quit, but if we hadn't, we'd have had to keep going, trying each new case from pairs of 24-factors with pairs of 30-factors: 2x12 with 5x6, 2x12 with 3x10, 4x6 with 2x15, 3x8 with 2x15, 2x12 with 2x15, 1x24 with 5x6, 1x24 with 3x10, 1x24 with 2x15, 1x24 with 1x30, 2x12 with 1x30, 3x8 with 1x30 and 4x6 with 1x30. And each one of these with the pair of different sign combinations.

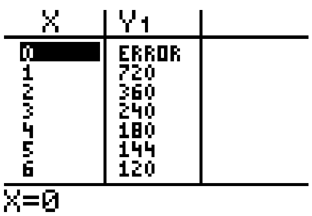
This is the main drawback of trial and error. It's time-consuming when there are a lot of options to check, and you have to make sure you get them all or you could miss the factoring combination you need.

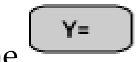
Example 13. $24x^2-71x-30$
We can do this same problem a little more quickly with factoring by grouping. The trick here is that grouping works on 4 terms and we only have 3. So what we are going to do is split up the middle term into two terms (that add up to our middle term) so that factoring by grouping will yield the result. We find this by what is sometimes called the ac-method. We are multiplying the leading coefficient (a) by the final constant (c) and factoring the result.

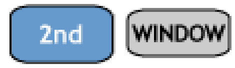
ac = (24)(30) = 720.

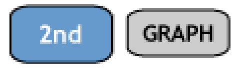
This is the one place where factoring by grouping can get tricky. We have to find the pairs of factors of a rather large number, 720 here. We can do this by factoring 720 into primes and methodically combining them, or we can use our graphing calculator to get





a list. I'm going to do the latter method. Go to the 

screen and enter the number you wish to factor divided by x as shown. The TBLSET screen 

should appear as shown to the left. Then click on 

to view the table. You can use the table to collect the pairs of factors of 720. Scroll down the list until the pairs start repeating in reverse. Use only pairs that are both whole numbers.

720 thus factors as 1x720, 2x360, 3x240, 4x180, 5x144, 6x120, 8x90, 9x80, 10x72, 12x60, 15x48, 16x45, 18x40, 20x36, 24x30, then it starts repeating in reverse. In order to split the middle term up correctly, we need factors of 720 that have a difference of 71. (We are using a difference of factors here because the last constant is negative.)

720-1= 719, 360-2=358, 240-3=237, 180-4=176, 144-5=139, 120-6=114, 90-8=82, 80-9=71, 72-10=62, 60-12=48, 48-15=33, 45-16=29, 40-18=22, 36-20=16, 30-24=6.

The pair that gives us the 71 we need is 9x80. Since we want to end up with a -71 when we subtract, we want 80 to be negative and 9 to be positive.

$$24x^2 - 71x - 30 = 24x^2 - 80x + 9x - 30$$

Now that we have 4 terms we can use factoring by grouping.

$24x^2 - 80x$ factors as 8x(3x-10)
9x – 30 factors as 3(3x-10)

$$24x^2 - 80x + 9x - 30 = 8x(3x - 10) + 3(3x - 10)$$

These have (3x – 10) in common, so factoring that out gives us:

$$8x(3x - 10) + 3(3x - 10) = (3x - 10)(8x + 3)$$

This is the same result we obtained using trial and error.

Both trial and error and factoring by grouping get you the same answer. Neither method is entirely without complications, but with practice both are effective. You do not need to know how to do both methods, just one. Choose the one you like the best and do enough examples to feel really confident with it.

Step 4. Check. Particularly in the beginning, and at any point where you've been away from factoring for a long time, you should check your work by FOILing your result to make sure it's getting you the correct answer. You should always also check all your factors for further factoring, including GCFs, difference of squares, etc. When the directions tell you to "factor completely" each factor should be either a GCF, a linear factor that has no common factors, or an unfactorable quadratic.

Example 14. $k^2 - 6k + 16$
We've seen some examples of things that aren't factorable. We call these polynomials "prime" by analogy with prime numbers. Sums of squares aren't factorable, nor is the quadratic that comes out of the sum/difference of cubes formulae. But we can encounter others.

In the example here, 16 factors as 1x16, 2x8 and 4x4. Because 16 is positive, we want these factors to add to become the middle 6. But when we add them we get only 17, 10 or 8. Since we have no way of obtaining 6 without subtracting (which we can't do and still get a positive 16) we say this polynomial is prime.

Be aware, while prime polynomials do occur, they are highly unlikely to appear more than once in 10-20 problems. As teachers, we want to test your factoring ability, and while recognizing that some polynomials can't be factored is important, doing it more than once (or on the outside twice) per problem set doesn't tell us much about your factoring skills. Use "prime" for your answer only when you are sure you've checked all other options thoroughly first. As far as guesses so, it's a bad one.

Practice problems.
Factor completely.

1. $12x^2 + 20x + 8$
2. $x^2 - 17x + 72$
3. $-16m^2n + 24mn - 40mn^2$
4. $64a^2 - 121b^2$
5. $36p^2 - 60pq + 25q^2$
6. $z^2 - 4z + 6$
7. $8p^3 - 1$
8. $x^6 + 4x^4 - 3x^2 - 12$
9. $4w^2 + 49$
10. $144 - 24z + z^2$
11. $100a^2 - 9b^2$
12. $m^3 + 4m^2 - 6m - 24$
13. $6t^2 + 19tu - 77u^2$
14. $4k^2 + 28kr + 49r^2$
15. $54m^3 - 2000$
16. $x^4 - 625$
17. $10r^2 + 23rs - 5s^2$

18. $ab + 6b + ac + 6c$
19. $56k^3 - 875$
20. $4p^2 - 26p + 40$