Betsy McCall

Graphing Systems of Inequalities

The first step in graphing systems of inequalities is to first be able to graph a single linear inequality (in two variables). There are two basic types of these:

1. Vertical and horizontal lines (with just one variable)
2. Linear inequalities with two (explicit) variables.

We also have two types of inequalities to concern ourselves with that will need to be represented on the graphs:

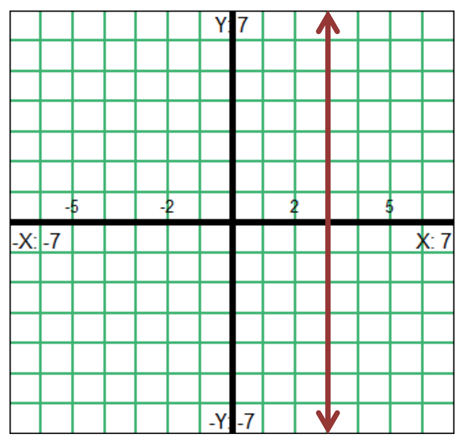
1. $\leq, \geq$ inequalities that contain equal signs
2. $<, >$ inequalities that do not contain equal signs.

To start with we will graph all our inequalities with the included equal sign because this is the most common type used in a system, but we will return to the second type later.

Once we have done these subsets of inequalities in two variables, then we will be able to tackle sets of them.

Example 1a. Vertical lines.

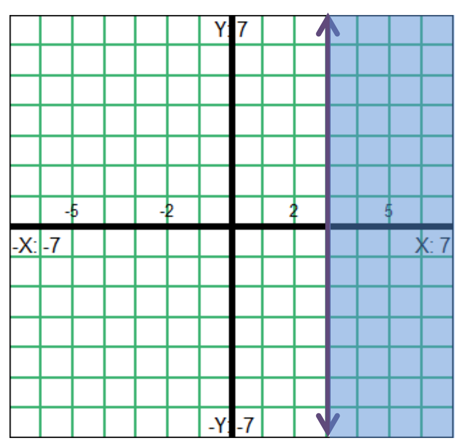
To graph an inequality, one first need to graph the boundary line that divides the half of the graph that satisfies the inequality from the half of the graph that does not satisfy an inequality. Thus, I will often refer to the type of inequality it is by referring to the properties of this boundary line. We first consider a vertical boundary line. Since vertical lines are represented by x=a, where a is a constant, we will be graphing inequalities of the form $x \leq a, x \geq a, x < a, x > a$. Consider the specific case of $x \geq 3$.



First, draw a line at x=3 as shown. The line is shaded solid because the inequality contains an equal sign.

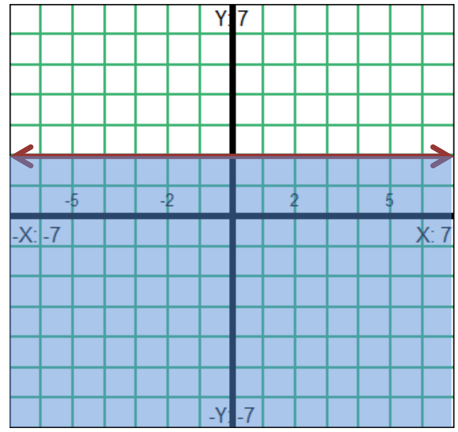
The second thing we need to do is to determine which side of the line satisfies the inequality. For lines with two variables, it can be useful to choose a point to test in the equation, but for vertical lines > or $\geq$, shade to the right of the line and for < or $\leq$, shade to the left of the line.

The final graph is shown below.



Example 1b. Horizontal lines.

Horizontal lines are done much the same way. These arise in inequalities that contain only the y variable, i.e. $y \leq b, y \geq b, y < b, y > b$.

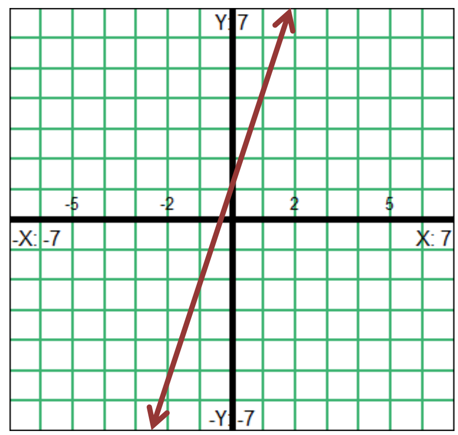


Let us consider the specific case of $y \leq 2$. Begin by drawing the boundary line, y=2. As before, the line is solid because the equal sign is included in the inequality.

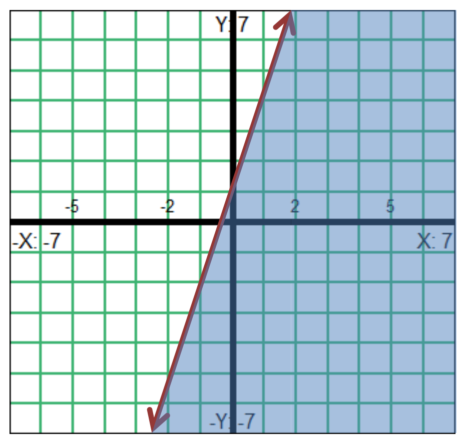
Then we need to shade on one side of the line or the other. For horizontal lines, > or $\geq$, shade above the line. For < or $\leq$, shade below the line. The final graph is shown here.

Example 2a. Lines solved for y.

If the line with two variables in it is already solved for y, determining the region to shade is easier than if the equation is not already so configured, so we will consider this case first.



Consider the inequality $y \leq 3x + 1$. This graph has a slope of 3, and a y-intercept at (0,1). We already have one point to plot for the line, we just need another. So, choose an x-value and find a second point. I choose x=1 and obtained the point (1,4) from the equation. On the graph I have drawn the required line. (Again, the line is filled solid because the equal sign is included in the inequality.)

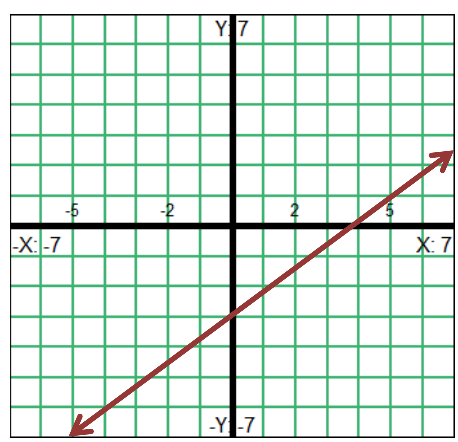


Now we need to determine which side of the line to shade on. The good news here is that we can use the same rule for this situation that we did for the horizontal line. Shade above the line (↑) if the inequality is > or $\geq$, and shade below the line (↓) if the inequality is < or $\leq$. Here, we have a less than inequality for y, so we will shade below the line.

This is precisely the trick our calculator's use when they do shading for us. We will talk about how to use the calculator's shading feature at the end of this section.

Example 2b-1. Linear equations in standard form.

If you have an inequality where the line is in standard form, you can plot the line easily from the intercepts, but deciding which side the line is on, is trickier.

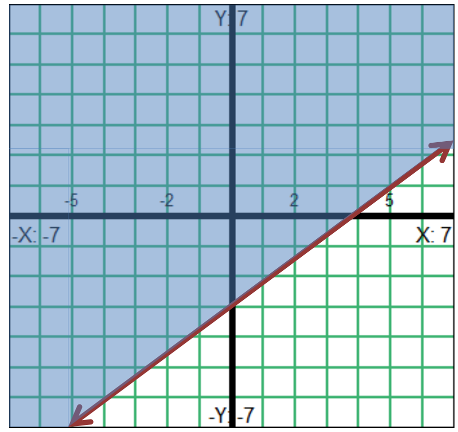


Consider the linear inequality $3x - 4y \leq 12$. We will first plot the line of equality that will divide our two regions.

The x-intercept, when y=0 is (4,0). And the y-intercept, when x=0, is (0,-3). The plotted line is shown to the right.

One strategy would be to solve the equation for y and use the method above. But, one potential pitfall here is that we may make an error in our algebra, and then we will get the wrong result.

Instead, we can work with the equation in the form it has now, by choosing a test point that we know is clearly on one side of the line or the other, and then check that point in the inequality to see if it's true or not. If it is true, we shade on the side of the line containing the point. If the inequality we get is false, then we shade on the opposite side of the line that does not contain the point. We must be careful, though, not to choose a point that is actually on the line, because this will give us no information.

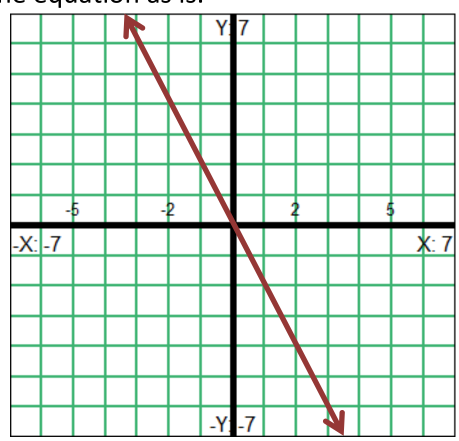


Typically, a useful point is the origin (0,0) since the arithmetic is especially easy, but beware if the line passes through the origin. In such a case, you will need to choose an alternate point. Here, however, (0,0,) works perfectly well since it's nowhere near the line.

Plugging (0,0) into the inequality, then, we get $3(0) - 4(0) \leq 12$ or $0 \leq 12$. Now we need to determine if 0 is actually less than (or equal to) 12. And it is, so the side of the line we need to shade is the side that contains the origin.

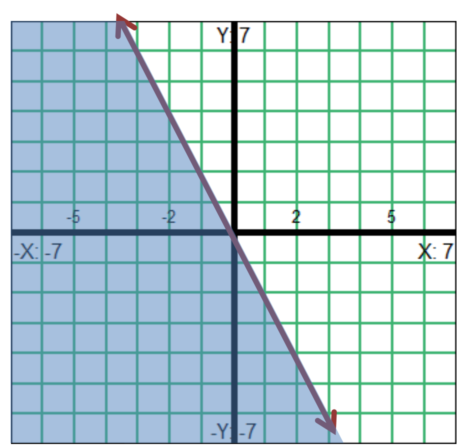
Example 2b-2. Line through the origin.

When the constant in the equation is a zero, we get a line through the origin, such as $2x + y \leq 0$. Here, we may find it advantageous to solve for y, but we can work with the equation as is.



The x-intercept, when y=0, also turns out to be the y-intercept, since x also equals 0 there. So one point on the line is the origin, (0,0). We need to find a second point to plot the line, so if I let x=1, I get the eqation $2(1) + y = 0$ or $y = -2$, and the point (1,-2). Plotting those points, we get the graph shown.

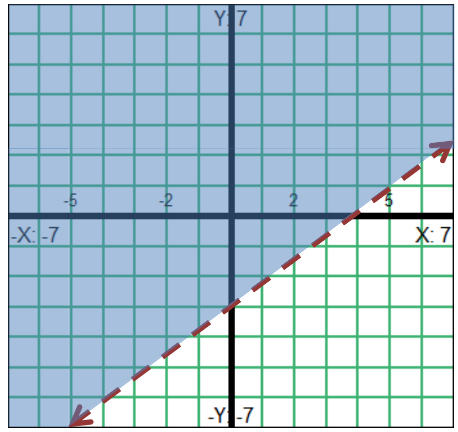
Since we can't use the origin as our test point, we need to choose another point not on the line. You can choose and point you like, but to make my arithmetic a bit easier, I like to choose a point on one of the axes, for instance, (2,0). This is clearly to one side of the line and so will give us the information we need. Plugging this into the inequality, we obtain $2(2) + 0 \leq 0$ or $4 \leq 0$. Is this expression true? No, since 4 is not less than (or equal to) zero. So we shade on the side of the line that does not contain the point (2,0), as shown in the second graph.



Example 3. Inequalities with no equal sign.

Before we go on and look at how to obtain these graphs in the calculator, and systems of multiple equations, we need to consider the case when our inequality does not include the boundary condition.

In such circumstances, our procedure for producing the graph is exactly the same as before with one exception: the line we draw is not included in the equality and so we will indicate that by using a dotted or dashed line when we graph it. Consider the inequality $3x - 4y < 12$. This inequality is very similar to the inequality in Example 2b-1.



The only difference in the drawing of the solution to the inequality here is the dashed line. We follow the rest of the procedures identically, including testing points.

You should watch out for these when you do your systems, in particular. Most systems will include the boundaries in both equations, but it's important to note when the boundaries are not included. Forgetting to notice the inequality type is an easy way to lose unnecessary points.

Example 4a. Graphing inequalities in the calculator.

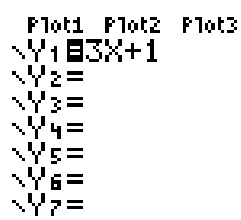
To graph inequalities in the Ti-83/84 calculator, there are some limitations.

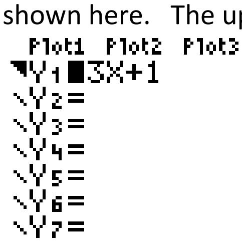
1. Vertical inequalities in x alone cannot be graphed on the calculator.
2. The calculator cannot distinguish between inequalities with equal signs and inequalities without them. You must do that yourself.
3. All equations must be solved for y, so be careful with signs (be sure to flip inequalities when working with division by a negative number—watch for negative coefficients of y), and also be sure that if you don't simplify all the way that you use the required parentheses.

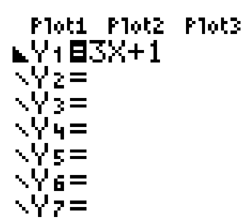
Since we already have a couple of graphs, let compare ones we have done by hand to ones the calculator will do so we have something to compare it with. We can start with the example in 2a: $y \leq 3x + 1$. This inequality is already solved for y, so it's a good first example.

Go to the 

screen and enter the equation for Y1 as shown. Then take your cursor and move to the left until it is under the line shown to the left of Y1. When you are there, the line will blink on and off. By hitting enter, you can scroll through the line options. Hitting Enter once will make the line thicker. Hitting it a second time gets you an upper triangle as shown here. The upper triangle represents the > (or $\geq$) condition. If you hit enter again, you will get a lower triangle shown to the right. This represents the < (or $\leq$) condition. Since we wish to graph $y \leq 3x + 1$, this latter is the condition we want.

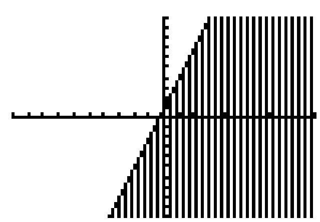




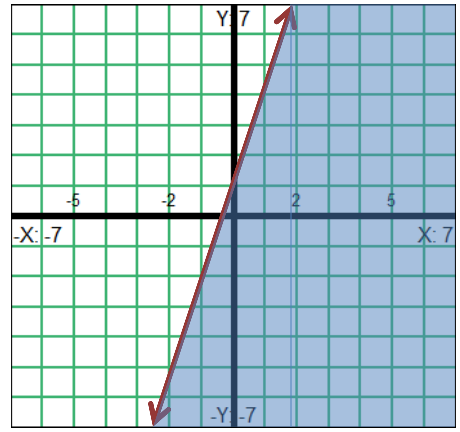


At this point, hit 

.



We obtain the graph as shown. Compare this with the graph obtained from the original Example done by hand shown for comparison again below.



The graphs are shaded on the same side.

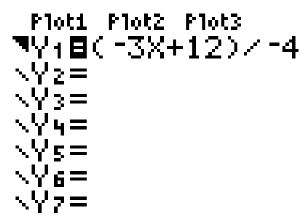
We can replicate the graph in Example 3 the same way.

Example 4b.

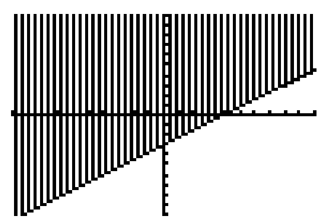
To graph the inequality $3x - 4y < 12$, we will need to solve this equation for y.

$$3x - 4y < 12$$
$$-4y < -3x + 12$$
$$y > \frac{-3x + 12}{-4}$$

Note that we had to flip the inequality when we divided by -4. And we will need to use parentheses unless we continue simplifying.



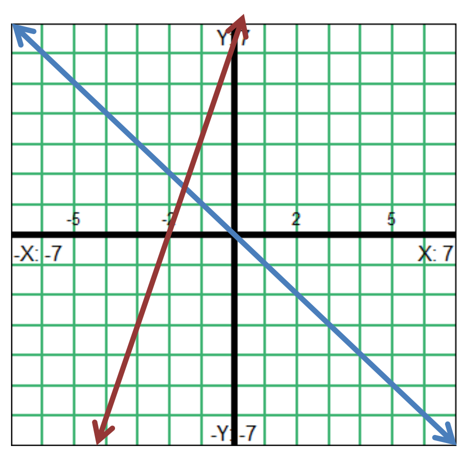
We will also need to press Enter several times on the graphing options in front of Y1 in order to scroll through the list back to the upper triangle. We need to use the inequality type in the solved equation. This is why it is very important to be careful with the signs, and a good idea to check a test point in the original equation if you have any doubt at all about making an error, even a careless one.



The final graph is shown here. Compare this graph with the one shown in Example 3. You'll note that the two examples we did do not treat the line any differently in the equality or the no-equality case.

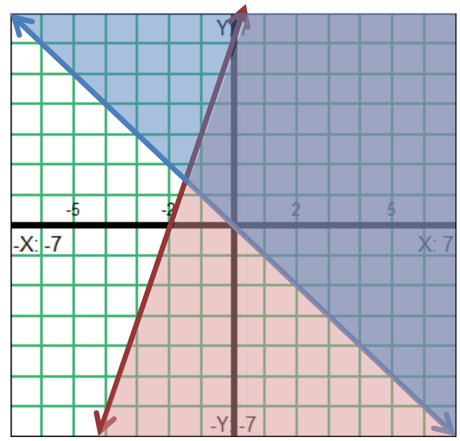
Let's move on to systems of inequalities. First, we'll do some examples with two inequalities at a time, and then we'll move on to systems with three or more inequalities, the kind you might find in the course of doing business analysis and linear programming.

Example 5. Systems of two inequalities.



Consider the system of inequalities $\begin{cases} y \geq -x \\ 3x - y \geq -6 \end{cases}$. Systems of inequalities generally assume the AND condition just like systems of equations. We want to know when both inequalities are satisfied simultaneously. (Later, we will consider an example when OR is used and how that differs.) Be must begin by drawing both equations on the same graph. As with the single inequality/equation examples, start with the intercepts and then move on to additional points if needed.

For the first line, both intercepts are (0,0). Substituting in x=1, we obtain a second point at (1,-1).



For the second line, when x=0, we obtain (0,6). And when y=0, we obtain (-2,0).

Plotting gives us the graph above. Both lines are included with the equal sign, so the lines we draw are solid. The first line is drawn in blue, and the second in the brick colour.

Our graph is divided up into 4 regions. One way to go about determining which region satisfies the system is to choose 4 points, one in each region to test in the system. If the point satisfies both inequalities, then the region that point is in is the region we should shade as our solution. Alternatively, we can graph each region for each line separately, and then look at where the two shaded regions overlap. This is the method we will use here.

Consider line #1. This is the inequality $y \geq -x$. This is the blue line going through the origin. Since the inequality is greater than, we will shade vertically above the line. Consider line #2. This line does not contain the origin, so let's test that point. $3(0) - 0 \geq -6$, or $0 \geq -6$. This is a true statement, so we shade on the side of the brick-coloured line containing the origin. When we do that, we get the graph above.

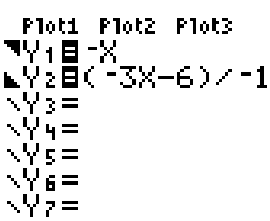
The purple region in the top right of the graph is the part that satisfies both equations, and so is the solution region. When doing problems on your own, you should clearly indicate that you understand this. This will be important when we compare to OR rather than AND systems.

Let us check by choosing a point in this region to see if it does indeed satisfy the system. Let's choose the point (5,0) on the x-axis. This is a point solidly in the twice-shaded region.

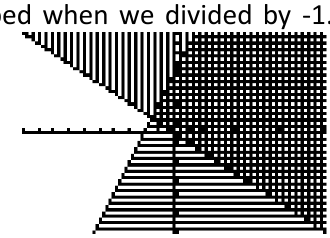
$$\begin{cases} 0 \geq -5 \\ 3(5) - 0 \geq -6 \end{cases} \text{ or } \begin{cases} 0 \geq -5 \\ 15 \geq -6 \end{cases}$$

Both of these inequalities are true: 0 is greater than -5, and 15 is greater than -6. So this is the correct region.

We can also graph two inequalities on our calculator. As in your Example 4's, solve the inequalities for y and enter them in the calculator as discussed in those examples. Enter the first equation as Y1 and the second as Y2 as shown.



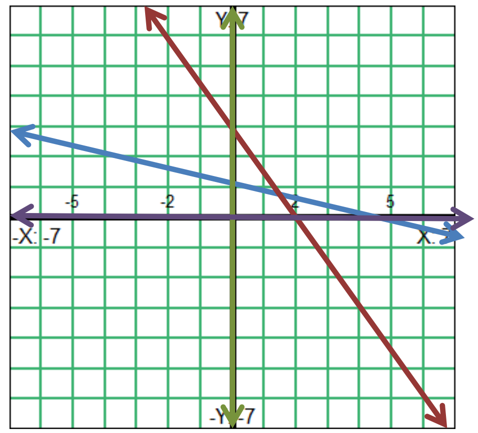
Notice that the inequality in equation two flipped when we divided by -1. The graph is shown to the right. The calculator uses shading hashes going in different directions so that you can see where the two regions overlap easily.



Example 6. A system with 4 inequalities.

When doing systems of equations that might possibly have applications in business, we often have many more inequalties than we have variables. Sometimes the extra inequalities are quite simple, but sometimes we really will have three or more separate lines. It becomes much harder to find the region satisfied by all the equations because the shading may actually interfere with interpreting the graph. Indeed, the calculator is almost impossible to read when three or more equations are used.

Consider a typical system of 4 linear inequalities: $\begin{cases} x + 4y \leq 4 \\ 2x + 3y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$.



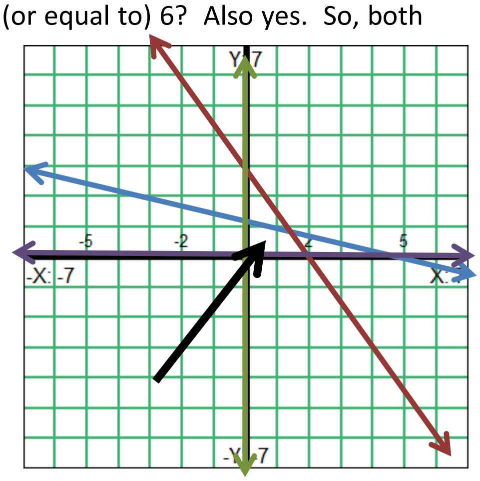
Begin as before by graphing all the lines. You'll notice that the last two inequalities are just saying that x and y should both be positive. When we do these on the calculator, we will account for these by adjusting the window to exclude anything that doesn't apply, but for now, we will graph them all.

The first equation has intercepts at (0,1) and (4,0). The second has them at (0,2) and (3,0).

The last two equations just restrict us to the first quadrant, so I will only concentrate on determining which side of the other two lines to shade on. Neither of these lines contains the origin, so let's test that point.

$$\begin{cases} 0 + 4(0) \leq 4 \\ 2(0) + 3(0) \leq 6 \end{cases}$$

So is 0 less than (or equal to) 4? Yes. What about 0 being less than (or equal to) 6? Also yes. So, both lines must be shaded below them in the direction of the origin.



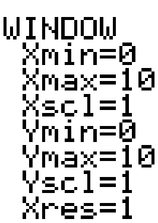
The region that we want, then is where the black arrow is pointing at is the region satisified by all 4 inequalities.

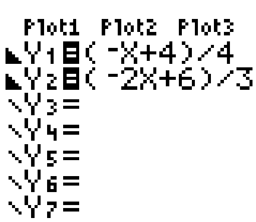
We can see this clearly on the calculator. First adjust your window to account for the vertical and horizontal restrictions. Set your YMIN to be 0 and you XMIN to also be zero. (Press the 

key to find this screen. Then put your remaining equations in the 

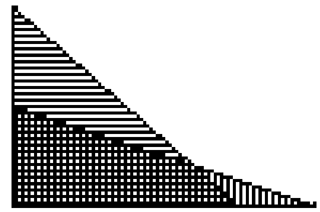
screen after solving for y. Both of these are going to be graphed using the lower triangle. Then select 

. It's difficult to see, so change your window so that XMAX is 4 (the value of the x-intercept on line #1), and YMAX is 3 (the value of the y-intercept on





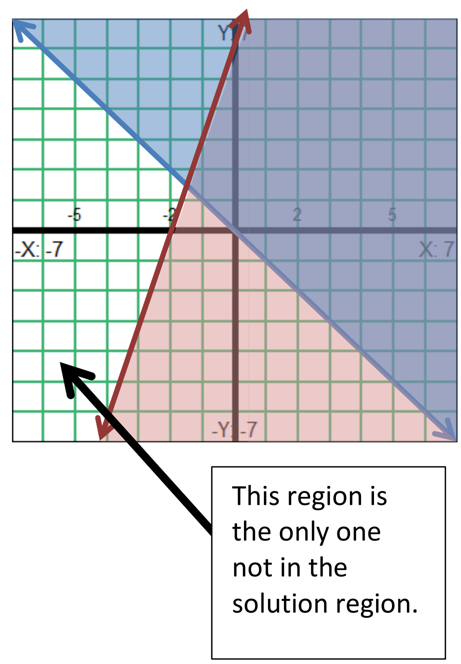
line #2). We use these values because they are the furthest away on the bounded region. You will have to experiment which window gives you the best graph. This twice-shaded region, bounded by the x- and y-axes, is the same region we obtained above.



Example 7. OR situations.

Systems of equations involving OR are uncommon, which is why we assume AND unless specified. The procedures are exactly the same, only the conclusion from the graph we obtain changes. Recall the system we graphed in Example 5: $\begin{cases} y \geq -x \\ OR \\ 3x - y \geq -6 \end{cases}$. We obtained the graph shown here.

In the AND situation, we said that we only wanted the region shaded from both inequalities. In the OR situation, we include **all** of the shaded regions. There is only one region excluded, the part with no shading. All the other parts of the graph satisfy at least one of the inequalities and so is therefore included in the OR solution.



Because of this difference it's especially important to indicate that you understand the difference and which region is not included, or is included.

Practice Problems.
Graph the solution to the systems of linear inequalities. Be sure to clearly indicate which regions satisfy the system.

1. $\begin{cases} 3x + y < 10 \\ -x + 5y \leq -10 \end{cases}$
2. $\begin{cases} 2x + y > -4 \\ x - y \geq 1 \end{cases}$
3. $\begin{cases} y \geq -x \\ 3x - y \geq -6 \end{cases}$
4. $\begin{cases} y < -2 \\ y < 2x + 3 \end{cases}$
5. $\begin{cases} x + y \geq -2 \\ x < 2y + 4 \end{cases}$
6. $\begin{cases} y \leq -3x + 4 \\ y \leq -2x + 3 \end{cases}$
7. $\begin{cases} 4x + 3y \leq 70 \\ x + y \leq 20 \\ y \geq 0 \\ x \geq 0 \end{cases}$
8. $\begin{cases} \frac{y}{2} - \frac{x}{6} \geq 1 \\ \frac{x}{3} - y \leq 1 \\ x \geq 0 \end{cases}$

9. $\begin{cases} x \geq \frac{5}{4}y - \frac{1}{2} \\ OR \\ 4x < 5(y+3) \end{cases}$

10. $\begin{cases} y \leq 2x \\ 5x - 2y \geq -20 \\ y \geq 0 \\ x \geq 0 \end{cases}$

11. $\begin{cases} y \geq -\frac{1}{4}x + 3 \\ OR \\ y \leq -\frac{5}{2}x + 2 \end{cases}$

12. $\begin{cases} 8x + 5y \leq 40 \\ y \leq \frac{1}{2}x \\ y \geq 0 \\ x \geq 0 \end{cases}$

13. $\begin{cases} y < -4x - 8 \\ y < -6x - 24 \end{cases}$