# Solving Systems of Linear Equations: Which Method to Use?

There are at least half a dozen methods that I know of for solving systems of linear equations, although in Algebra, we typically only learn three of these methods at first: solving by substitution, solving by elimination, and solving by graphing. All these methods work quite well for systems of two variables (though graphing is problematic in three dimensions and impossible with more than three variables). Once we've learned all three methods, how do we know which method to use when we are given the choice? What makes one method better than another in a particular situation?

It should be noted that all three methods illustrated will work with any system of two variables. What we are seeking is some way of determining which method will be easiest to use in a given situation. Choosing a method you feel comfortable with is one way of avoiding errors. Choosing a method that tends to avoid situations where errors are common is another way of choosing, and that is what we will be illustrating in this handout. Your level of comfort with a particular method is something the individual student must measure.

Let us begin by illustrating each method in turn, and listing advantages and pitfalls, and then we will compare and contrast the methods directly at the end of the handout.

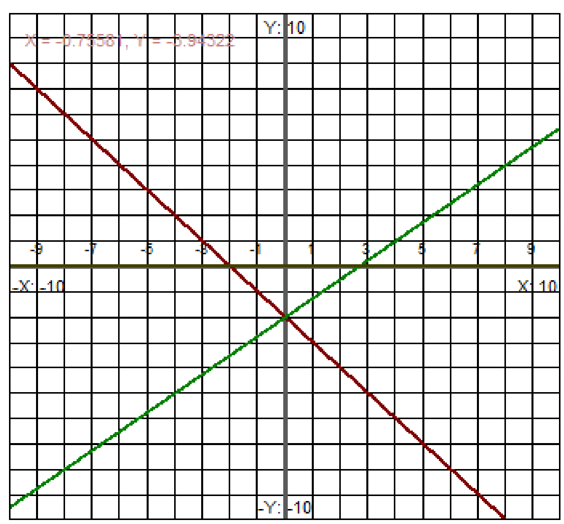
**Example 1**. Graphing.

Solve the system of equations given by $\begin{cases} x + y = -2 \\ 3x - 4y = 8 \end{cases}$ using the graphing method.

To solve the system by graphing, the first thing we must do is solve both equations for y.

$$\begin{cases} y = -x - 2 \\ y = \dfrac{3}{4}x - 2 \end{cases}$$

Then we can enter these equations into our graphing software. Graph the first equation as Y1, and the second as Y2.




The graph we obtained is shown here. We can see from the graph where the point of intersection is, and in this case, read off the graph that the point of solution is (0,-2).

**Potential Advantages of solving a system of equations Graphically**.

- If your equations are already solved for y, they can go directly into your calculator/graphing program.
- If the equations have integer solutions for both variables, they are very easy to read off the graph, particularly if you are displaying gridlines as we are here.

- If you are going to round your answer anyway, and your graphing program has a feature to find the intersection for you, then you can get the required accuracy directly from the program without algebra.
- If your system of equations is quite complicated with many fractions, you do not need to fully reduce for the graphing method to work.
- If the system is inconsistent, the graph of two parallel lines is quite distinct (with care).
- This method works for two variable, nonlinear systems as well.

**Potential Disadvantages of solving a system graphically.**

- If the equations are not already solved for y, you will have to do this algebra step first.
- If you don't reduce your fractions first, as I did in the example, you will need to be very careful about the use of parentheses when you enter the equation. Leaving out required ones will result in graphing the wrong equation and obtaining a solution to a different system than the one you wanted. For instance, without reducing my Y2 would have to have been Y2=(-3x+8)/(-4). Without the set of parentheses in the numerator, the 8 would have been the only thing divided by the -4.
- If you need exact answers and the fractions that result at an intersection are uncommon, your calculator may not be able to give you enough decimal places to make the conversion back to fractions on the main screen.
- You may graph two equations and only see one equation on the screen. You must be careful to ensure that the second equation is not just simply out of range of the window you are currently using before concluding the solution is an infinite set. Similarly, if you don't see an intersection on screen, it may not be because the lines are parallel, but because the intersection is far out of the window. You will need to adjust the window to find the intersection and get it on the screen before your intersect feature will find it. You may falsely interpret lines with very similar slopes as being parallel when they are not quite.
- If there are more than two variables, and certainly more than three, this method is more difficult (you will need a three-dimensional grapher), or impossible (we cannot see in four or higher dimensions).

**Example 2**. Substitution.

Solve the system of equations given by $\begin{cases} x + y = -2 \\ 3x - 4y = 8 \end{cases}$ using the substitution method.

To solve the system by substitution, I need to choose one of the equations and solve for one of the variables. I will choose the first equation and solve for y as we did in example 1.

$$y = -x - 2$$

Now I use this information to replace y in the second equation. Instead of $3x - 4y = 8$, I get

$$3x - 4(-x - 2) = 8$$

This gives me a single equation with just one variable, which I can then solve.

$$3x + 4x + 8 = 8$$

Adding 8 to both sides and combining like terms gives:

$$7x = 0$$
$$x = 0$$

Once we know that x=0, we can substitute this value back into our modified first equation to obtain the solution for the second variable.

$$y = -(0) - 2$$
$$y = -2$$

Thus, the ordered pair solution to our system is (0,-2). (This is the same solution we obtained in Example 1.)

**Potential Advantages of solving systems by Substitution**

- If at least one of the variables in one of the equations is an understood 1 or -1, you can avoid introducing any fractions into your algebra.
- This method works on systems with any number of variables, and on linear and non-linear systems.
- You only need to manipulate one equation with two variables, and just long enough to solve for one variable. The process of substitution reduces the second equation to a single variable problem (or at least reduces it by one variable), and so it is easier to do algebra on it.
- This is often the first algebraic method we learn, and so students often feel the most comfortable with it.
- It is capable of giving exact answers, when exact answers are required.

**Potential Disadvantages of solving systems by Substitution**

- If none of the variables has a coefficient of 1 or -1, you will have to work with the fractions you introduce into the system by solving for a variable.
- When there are more than two variables this procedure can be tedious since you can only eliminate one variable at a time. For instance, in a three-variable system you will need to substitute to eliminate one variable in each of the remaining two equations making them two variable equations. And then do substitution again on the remaining two equations now in two variables to get down to a single equation with one variable that can be solved, finally.
- It is often the case that students solve for one variable, and then to forget to solve for the second one.

**Example 3**. Elimination (by Addition)

Solve the system of equations given by $\begin{cases} x + y = -2 \\ 3x - 4y = 8 \end{cases}$ using the elimination method.

To employ the elimination method, we to multiply one or both equations by a constant to that we can match up the size of one of the variables in one equation with the size, but opposite sign, of the same variable in the other equation. When we add then the same size opposite sign terms will cancel out. Be aware that for this to work correctly, the variables must be on the same size of the equal sign when we add them, so if they are not lined up as they are in this example, you must do that first.

I prefer to start this method looking for two variables that already have opposite signs so that I don't have to worry about working with negatives. In this set of equations, the y variable in the first equation is positive, and the y variable in the second equation is negative, so I'm going to work with those. How can I get them to match? Since the coefficient of y in the first equation is 1, I can multiply it by the coefficient (without the sign here) from the second equation to make them equal. In another system, it may be necessary to multiply both equations, but not in this example. Remember to multiple the ***whole equation*** by the number you want to modify y by or you will change the equation, and thus, the system. After multiplying the first equation by 4, I get the following.

$$\begin{cases} 4x + 4y = -8 \\ 3x - 4y = 8 \end{cases}$$

Now, add vertically.

$$\begin{array}{c} 4x + 4y = -8 \\ \underline{3x - 4y = 8} \\ \\ 7x = 0 \end{array}$$

So, we can solve the single equation for the one variable that remains: x=0. Then we have to plug this value back into either of the other two equations to obtain the solution for the second variable. Choose one of the original equations (you are more likely to catch an error this way), and I suggest choosing one that requires the least algebra to solve even though, in principle, it doesn't matter which you use.

$$\begin{array}{c} 0 + y = -2 \\ y = -2 \end{array}$$

Thus the solution to the system is the same obtained by the other methods, (0,-2).

**Potential Advantages of solving systems by Elimination**

- The entire procedure can be done by working only with whole numbers, so if you are weak on fractions, you can avoid them.
- This method is a little easier than the others for working with systems of equations with more than two variables, and can be generalized to another method of solving equations, matrices, that you may learn later on.
- The algebra after the elimination step will be further along than at the same point in the substitution procedure. There will be fewer steps left, so it is a bit more efficient.
- Like substitution, it can provide exact answers when they are needed.

**Potential Disadvantages of solving systems by Elimination**

- Sometimes students find it confusing to match coefficients when both equations need to be modified. Students may also be weak on it because it is often the last method that is learned of the three.
- If you commonly make errors when distributing negatives, you will need to be especially careful when you must change sizes in elimination.
- In order to avoid working with fractions, you may be required to work with and reduce very large numbers.

**So how do I know which method to use?**

Of course, they all work, and so some of the choice is a matter of preference when you are given the option to choose your own method. But these are some guidelines that I look for when I have to make a choice:

1. If the system already has both equations solved for y, and there are only two variables, use **graphing** as a first choice. Fall back to substitution if you can't get an exact solution from the calculator and need one.
2. If the system of equations requires a lot of algebra steps to simplify each equation, using **graphing** to avoid reducing, as long as you can solve for the y variable without too much difficulty.
3. If the system has at least one equation solved for either variable, use **substitution**.
4. If a system with more than two variables has any equations with fewer variables, this system will be easier than usual for **substitution**. Use your best judgment.
5. If a system of equations has at least one variable with a coefficient of 1 or -1, use **substitution** by solving for that variable.
6. If none of the coefficients in the equation is 1 or -1, use **elimination**.
7. If you have large numbers of variables, use **elimination** except possibly in certain special cases.
8. In problems where you are allowed to choose your own method, do not be afraid to switch methods in the middle of a problem. If you see an easy way to substitute for the first step, but then obtain equations that have no coefficients of 1 or -1, it's okay to switch to elimination (or graphing when you are down to two variables) to continue solving the system. Just don't forget to go back and solve for all the original variables.
9. When in doubt, use the method that you feel most comfortable with, and that, in your experience, you are most likely to get correct.

**Practice Problems.**
For each of the systems below, explain which method you would use and why. Then solve the system. State your solution as an ordered pair (or ordered triple). Your answer should discuss the merits of the method you chose, *not just because it's the only one you know*.

a. $\begin{cases} x + 2y = 4 \\ x + 1 = 5 - 2y \end{cases}$

b. $\begin{cases} 3x - 2y = -2 \\ 2x + y = 8 \end{cases}$

c. $\begin{cases} y = x + 2 \\ 3x - 3y = -6 \end{cases}$

d. $\begin{cases} y = 3x - 1 \\ y = -2x + 5 \end{cases}$

e. $\begin{cases} -6x - 2y = 4 \\ 5x + 3y = -2 \end{cases}$

f. $\begin{cases} x + 2y = 4 \\ x + 4y + z = 11 \\ 3x + 3y - z = 5 \end{cases}$

g. $\begin{cases} 2x + 2y + 5z = 9 \\ -3x + 4y + 2z = 13 \\ 3x + 5y - 7z = 25 \end{cases}$