Solving Exponential and Logarithmic Equations

There are several techniques we can use for solving exponential and logarithmic equations. Which technique we use depends on the equation.

1. **Eliminate the "base".**
   a. We can use this technique with both types of equations, but it can only be used when the bases can be made the same. For example.
      i. $3^{x+1}=9^x$
         In this example, 9 is a power of 3: 3^2, and so we can write the problem with the same base. $3^{x+1}=(3^2)^x$, which simplifies to $3^{x+1}=3^{2x}$, and now that the bases are the same, we can drop the 3's and solve: x+1=2x.
      ii. $4^{x+3}=8^{x-1}$
         Here, both bases must be converted to powers of 2: $(2^2)^{x+3}=(2^3)^{x-1}$, which simplifies to $2^{2x+6}=2^{3x-3}$, and now we can drop the bases and solve 2x+6=3x-3.
   b. There is a similar procedure that applies to logarithmic equations as well.
      iii. $\log_3 x=\log_3(2x-8)$
         When the bases of the logs are the same, they can be dropped: x=2x-8.
      iv. $\ln x+\ln(x-1)=\ln 6$
         You may need to apply log properties before drop the logs. These problems must be in a very specific form: $\ln[x(x-1)]=\ln 6$, and now drop the logs to solve: $x^2-x=6$.

   Note: for log problems, be sure to check that your solutions are valid!

2. **Quadratic form.**
   a. These problems look rather similar to the "eliminate the base" problems, but contain three (or more) terms. For quadratic form, there will be two exponential terms and one constant, and either one base will be the square of the other exponential term, or else the exponent will be twice the other exponent. These types of problems can be generalized to any size polynomial, but we will not encounter them in this course.
      i. $3^{2x}+3^x-2=0$
         Here, since $3^{2x}=(3^x)^2$ we substitute for the base 3^x=y to obtain $y^2+y-2=0$. Solve from here. Once we find y=1,-2, we have to put these values back into our substitution to solve for the original variable. Some values may need to be discarded since $3^x\neq -2$ for any value of x.
      ii. $36^x-6\cdot 6^x+9=0$
         Here, since 36 = 6^2, we can still apply our substitution y=6^x to obtain $y^2-6y+9=0$.
      iii. $2^{2x}+2^{x+2}-12=0$
         This is another interesting variation on the same type of problem. Here, the coefficient has been absorbed into the middle term. Since $2^{x+2}=2^2\cdot 2^x=4\cdot 2^x$, that extra +2 does not really change the technique used. We obtain the equation $y^2+4y-12=0$.

iv. $3^x - 14 \cdot 3^{-x} - 5 = 0$

To solve this equation, we have to first get it into quadratic form, since we can't do the same type of substitution here that we did before. However, if we multiply the whole equation by 3^x, it will look more familiar: $3^{2x} - 14 - 5 \cdot 3^x = 0$, and now substitution will yield $y^2 - 5y - 14 = 0$.

3. Mixed bases.

a. Some exponential problems cannot be written with common bases. Here, we will use a standard log function (either natural log or $\log_{10}$) and apply log properties to isolate the variable. Sometimes, this can produce nasty results, depending on the complication level of the problem.

i. $e^{x+3} = \pi^x$ or $2^{x+3} = 5^x$

Both these problems can be solved the same way, using the same log functions. The first suggests natural log since it will cancel nicely with the *e* on one side, however, the same log, and the same steps applies to the second version. For the first one: $\ln(e^{x+3}) = \ln(\pi^x)$. Apply the log property for exponents to obtain: $(x+3)\ln(e) = x\ln(\pi)$. Natural log of any constant is just a number. If the number is not 1, as it is here on the left, distribute and collect like terms as you would for any other constant. $x + 3 = x\ln(\pi) \to x - x\ln(\pi) = -3 \to$ $x(1 - \ln(\pi)) = -3 \to x = \dfrac{-3}{1 - \ln(\pi)}\, or\, \dfrac{3}{\ln(\pi) - 1}$.

ii. $0.3\left(4^{0.2x}\right) = 0.2$

If the exponential is not by itself, you can't apply log rules first thing. First, isolate the exponential, and then follow the steps as before.

iii. $1500 = 750\left(1 + \dfrac{0.06}{12}\right)^{12t}$

Financial models are does like this as well, when you are solving for the variable in the exponent. The base here is $\left(1 + \dfrac{0.06}{12}\right)$, which is just a constant. Apply the natural log to both sides and use log rules to isolate the variable.

iv. $\log_2(x) = \log_5(x)$

Log problems with different log bases can also be transformed into like logs so that a solution can be obtained. We do this by using the change of base formula: $\log_b(a) = \dfrac{\ln(a)}{\ln(b)}$. Applying this to both sides we get: $\dfrac{\ln(x)}{\ln(2)} = \dfrac{\ln x}{\ln 5}$

Cross-multiply: $\ln(5)\ln(x) = \ln(2)\ln(x) \to \ln(x)^{\ln 5} = \ln(x^{\ln 2})$. Drop the outside natural logs: $(x)^{\ln 5} = x^{\ln 2} \to x^{\ln 5} - x^{\ln 2} = 0$. This involves some pretty fancy factoring, but it factors as: $x^{\ln 2}(x^{\ln\left(\frac{5}{2}\right)} - 1) = 0$, and from this we can obtain solutions of x=1 or x=0. Some of these kinds of problems CAN be done algebraically, but most of them are probably easier to do with the last technique.

4. Special Cases.

a. Some problems can't be solved algebraically and must be done graphically. These problems include those with a mixture of constant log bases and variable log bases, as well as those with a mixture of algebraic and logarithmic term, or a mixture of algebraic and exponential terms. Examples include: $\log_3(x+4)=\log_x(10)$, $e^x = x$, $x^2=\log(x)$, $e^{-x}=-\ln(x)$, $\log_2(x-1)-\log_6(x+2)=2$. In these problems, it is impossible to isolate the variable, and so impossible to solve without numerical methods, such as the calculator provides. For the non-standard log-bases, you will need to apply the change of base formula shown previously to get them into your calculator.

i. $\log_3(x+4)=\log_x(10)$

Rewrite as $\dfrac{\ln(x+4)}{\ln(3)}=\dfrac{\ln(10)}{\ln(x)}$. Graph (shown to the right). Use the INTERSECT feature on your calculator to approximate the solution. Carefully note how many decimal places accuracy the problem is requesting. Do not attempt to approximate the solution by using the TRACE feature as it won't be accurate enough.