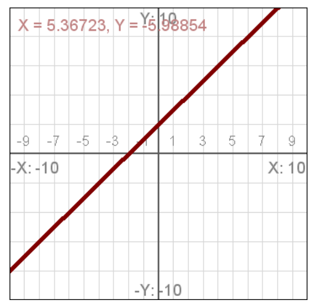
Betsy McCall

Rational Functions

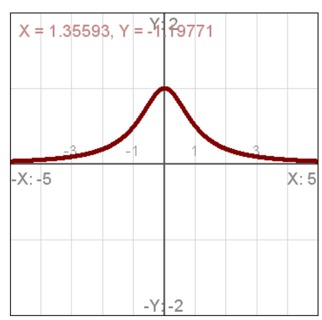
We will look at three types of rational functions: 1) those with no vertical asymptotes, 2) those with one vertical asymptote, 3) those with two vertical asymptotes.

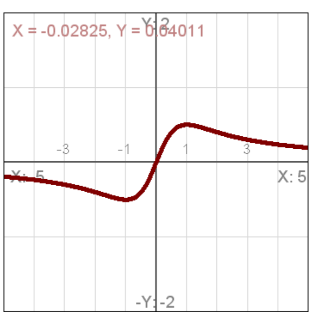
1. **No vertical asymptotes**.
   A rational function has no vertical asymptotes when at least one of two conditions applies:
   - Any factors that could be equal to zero will cancel with the numerator (you will have a ***hole*** for each unique, cancelling factor)
   - Any non-cancelling factors have complex solutions and cannot be made equal to zero with a real number

   a. Example. Graph with a hole. $R(x)=\dfrac{x^2-x-6}{x-3}$. Here the numerator factors as $(x+2)(x-3)$, allowing the $x-3$ factor to cancel, reducing the function to a nearly equivalent function $f(x)=x+2$. It is "nearly equivalent" because R(x) is not defined at x=3 (i.e. it's domain is $(-\infty,3)\cup(3,\infty)$), whereas f(x) is defined for all real numbers. The graphs will appear identical, but R(x) will have a single point of discontinuity, at x=3 (at the point (3,5)), which we should indicate on any graph. The calculator will not show this discontinuity, but you will see with ERROR message if you go to the Table. This graph is sometimes describes as having an oblique asymptote, although the "oblique asymptote" is identical with the graph. A common characteristic of rational functions with only holes and no vertical asymptotes is that when division (long or synthetic as appropriate) is performed, a remainder of 0 is obtained.



   b. Example. Graph with complex factors. $R(x)=\dfrac{1}{x^2+1}$. Here, the denominator does not factor in real numbers any further, and so it cannot be equal to zero. There is a horizontal asymptote at y=0 (since the degree of the denominator is greater than the degree of the numerator), however, there is no vertical asymptote. The maximum is found at x=0, or (0,1). Variations on this would include $R(x)=\dfrac{x}{x^2+1}$, below. This graph also has a horizontal asymptote at y=0, for the same reason as the first graph, but this one has an x=intercept at x=0.





Use the information here to graph the following functions, by hand. Be sure to look for the domain, any oblique or horizontal asymptotes, holes, x- and y-intercepts.

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i. $R(x)=\dfrac{2}{x^2+4}$

ii. $R(x)=\dfrac{x-1}{x^2+3}$

iii. $R(x)=\dfrac{x^3+8}{x+2}$

iv. $R(x)=\dfrac{x^2+x-2}{x^3-x^2+x-1}$

v. $R(x)=\dfrac{x^3+7x^2-4x-28}{x^2-4}$

vi. $R(x)=\dfrac{x^2}{x^2+1}$

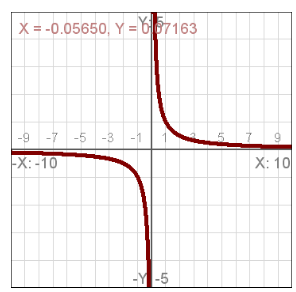
vii. $R(x)=\dfrac{x^3}{x^2+1}$

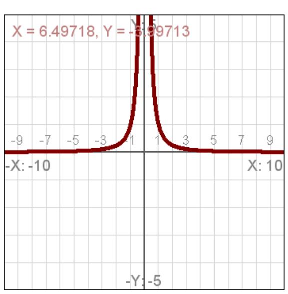
viii. $R(x)=\dfrac{x-1}{x^3-1}$

ix. Explain the relationship between (vi) and $S(x)=1-\dfrac{1}{x^2+1}$. How does this help you graph the equation in (vi)?

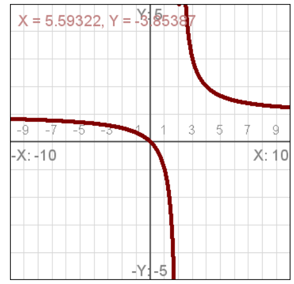
x. Explain the relationship between (vii) and $T(x)=\dfrac{x}{x^2+1}$ from the example? How does this help you graph the equation in (vii)?

2. **One vertical asymptote**.
   a. Example. More "typical" of rational functions that you will encounter will be those with one or more vertical asymptotes. The simplest example is in our library of functions $f(x)=\dfrac{1}{x}$ whose graph is shown here. You will notice that the vertical asymptote occurs at x=0, where the denominator would be equal to zero. As the graph crosses the line x=0, since the factor in the denominator is linear (or really, any odd power), the graph crosses from negative ∞ to positive ∞ on the other side. This is to be expected for all such asymptotes.

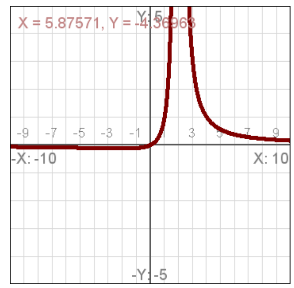




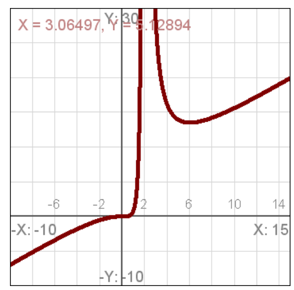
   b. Example. Consider, then, in comparison, the graph of $f(x)=\dfrac{1}{x^2}$ from our extended library of functions. In this case, the graph has a vertical asymptote also at x=0, but as it crosses from negative x's into positive ones, no sign change in the graph occurs. This happens because the factor in the denominator is squared and so the result will always be the same sign (similar behaviour will be seen for other even powers). This behaviour depends strictly on the degree of the factors at the asymptote and nothing else about the function. What the numerator can change is whether the graph is approaching positive or negative ∞. Both of these graphs have horizontal asymptotes at y=0, but that behaviour does depend strictly on what the numerator is doing relative to the denominator.



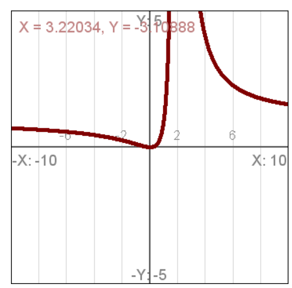
c. Example. $R(x)=\dfrac{x}{x-2}$ is shown to the right. Here, the vertical asymptote is at 2, with an x-intercept at x=0 (0,0), and a horizontal asymptote at y=1 because the numerator and the denominator have the same degree.



d. Example. $R(x)=\dfrac{x}{(x-2)^2}$. Compare that graph to the one below it. The only difference here is the square added to the denominator. The degree of the numerator is now less than the degree of the denominator, so the horizontal asymptote is at y=0, but there is still an x-intercept at x=0 (0,0), and a vertical asymptote at x=2. There is no sign change, however, now, as the graph crosses that line.



e. Example. $R(x)=\dfrac{x^3}{(x-2)^2}$. Here, the graph has an oblique asymptote, although the other features of the previous graph remain the same. Oblique asymptotes can often by difficult to capture on the standard window. This one is shown on [-10,15]x [-10,30] to capture the upper portion of the graph. Because the asymptote does not switch signs, the minimum of the second half without the intercept is not obliged to ever approach zero.



f. Example. Recall that the degree of the zero matters for its behaviour. The last graph shown is $R(x)=\dfrac{x^2}{(x-2)^2}$. Remember that for zeroes with an even power, the sign of the graph remains the same on both sides of the zero, and as here, deflects off the axis, touching it rather than passing through it.

Use the information here to graph the following functions, by hand. Be sure to look for the domain, any oblique or horizontal asymptotes, vertical asymptotes, holes, x- and y-intercepts.

i. $R(x)=\dfrac{1}{x-3}$

ii. $R(x)=\dfrac{x+1}{x-4}$

iii. $R(x)=\dfrac{x-1}{x^2-2x+1}$

iv. $R(x)=\dfrac{x+1}{x^2-2x-3}$

v. $R(x)=\dfrac{x^2-5x+6}{x^2-2x-3}$

vi. $R(x)=\dfrac{x^2-4x+4}{x-3}$

vii. $R(x)=\dfrac{x^2+1}{x-2}$

viii. $R(x)=\dfrac{2x-7}{3x+5}$

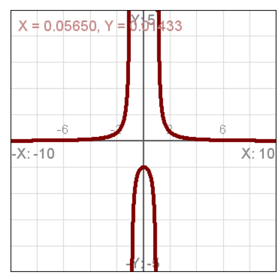
ix. $R(x)=\dfrac{x^2-7x+12}{(x^2+1)(x-1)}$

x. $R(x)=\dfrac{x-5}{(x+2)^2}$

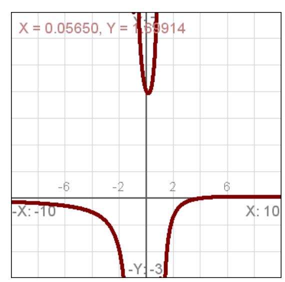
xi. Does the graph of the function in (iii) have a hole or not? Why or why not?

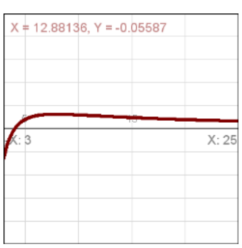
3. **Two vertical asymptotes**.
Rational function graphs can have any number of vertical asymptotes, depending only on the number of factors with real solutions in the denominator, but the behaviour between any pair of asymptotes can be seen by examining graphs with only two of them.

a. Example. $R(x)=\dfrac{1}{x^2-1}$. This denominator factors into two linear factors, $(x-1)(x+1)$, creating two vertical asymptotes. The numerator has no zero between the two asymptotes, and so, the graph as it approaches the asymptote must stay either positive or negative. Because the graph must be positive for large values of x (see the behaviour of $f(x)=x^2-1$ for comparison), the graph approaches the horizontal asymptote y=0 from the positive side of the x-axis. Thus, when it hits the vertical asymptote, the graph is forced to flip from positive infinity to the negative infinity on the other side.

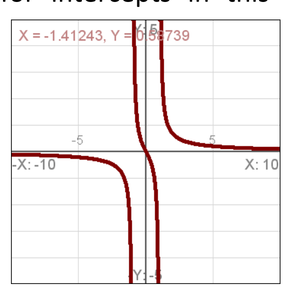


b. Example. $R(x)=\dfrac{x-4}{x^2-1}$. There are two ways to introduce x-intercepts into rational graphs with two vertical asymptotes: 1) outside both asymptotes, or between them. In this example, the intercept is to the right of both asymptotes. If we zoom in on the intercept, we see that the graph actually overshoots the asymptote, and then slowly approaches it again. This is typical for intercepts in this position.



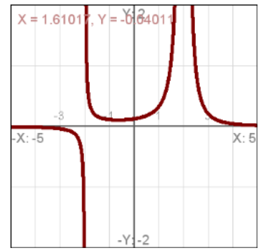


c. Example. $R(x)=\dfrac{x}{x^2-1}$. In this case, the intercept has been introduced between the two vertical asymptotes. We no longer have pseudo-parabolic behaviour, but something more line pseudo-cubic behaviour as the graph crosses the intercept. As with the

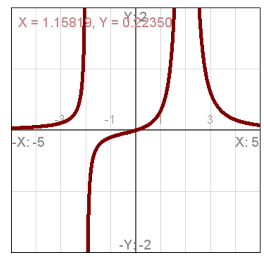


previous case, the long-run behaviour is now "odd" in the sense that the graph approaches the horizontal asymptote from below on the left, and from above on the right. Each of the vertical asymptotes remains linear, and so the behaviour of the graph as it crosses each asymptote is to switch from negative ∞ to positive, or vice versa.

d. Example. $R(x)=\dfrac{1}{(x-2)^2(x+2)}$. When we considered the case of the single vertical asymptote, we considered the possibility that the single linear factor might be repeated. That can occur in the two asymptote case as well. Here, the factor, corresponding the asymptote at x=2 is squared. As with the graph of $f(x)=\dfrac{1}{x^2}$, the graph does not change signs as it crosses the asymptote. However, for the linear factor at x= -2, the graph does change signs here. There are no x-intercepts between the two asymptotes, and so the graph remains entirely on one side of the axis in that region.



e. Example. $R(x)=\dfrac{x}{(x-2)^2(x+2)}$. If we introduce an intercept in between the two asymptotes, we see that pseudo-cubic behaviour again as the graph crosses the intercept. Notice that the behaviour at the intercepts remains the same: the linear one switches sign, while the squared one stays the same sign on either side.



i. What are some other possible variations on the two asymptote graph? Match the graph to the function.

a. $R(x)=\dfrac{1}{(x-2)^2(x+2)^2}$

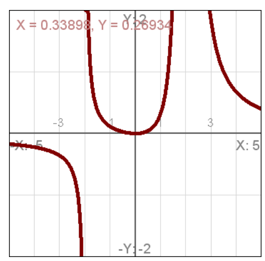
b. $R(x)=\dfrac{x^2}{(x-2)^2(x+2)}$

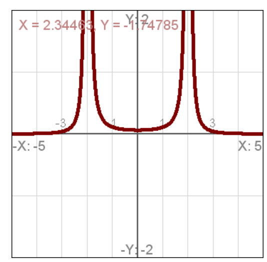
c. $R(x)=\dfrac{x-4}{(x-2)^2(x+2)}$

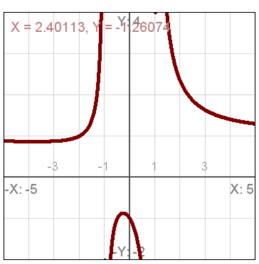
d. $R(x)=\dfrac{x^2-1}{(x-2)(x+2)^2}$

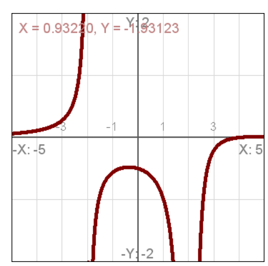
e. $R(x)=\dfrac{x^3-1}{(x-1)^2(x+1)}$

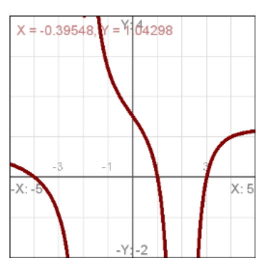
f. $R(x)=\dfrac{(x-1)(x-3)(x+4)}{(x-2)^2(x+2)}$

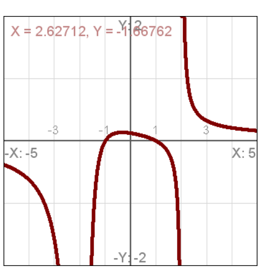












ii. Use the information here to graph the following functions, by hand. Be sure to look for the domain, any oblique or horizontal asymptotes, vertical asymptotes, holes, x- and y-intercepts.

a. $R(x)=\dfrac{x-1}{(x-4)^2(x+2)}$

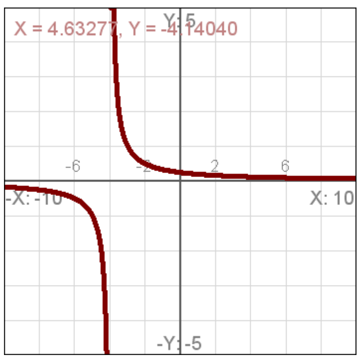
b. $R(x)=\dfrac{(3x+4)(2x+1)}{(x-2)(x+5)}$

c. $R(x)=\dfrac{x^2-1}{(x^3-1)(x^2-4)}$

d. $R(x)=\dfrac{x^3+8}{(x-3)(x+4)}$

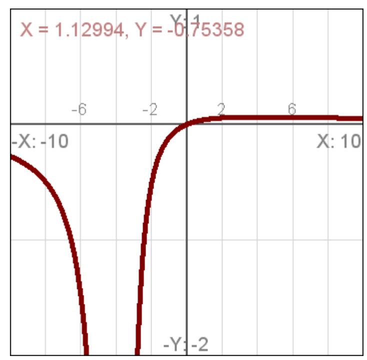
iii. For each of the graphs below, find a function that represents the graph. (There may be more than one acceptable answer.) To help you, you will need any horizontal or oblique asymptotes, any vertical asymptotes, and intercepts. (You may assume the graph has no holes, and that all relevant values are whole numbers.)

a.

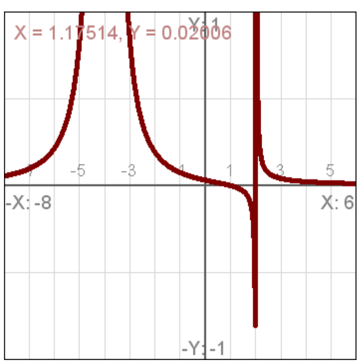


a.

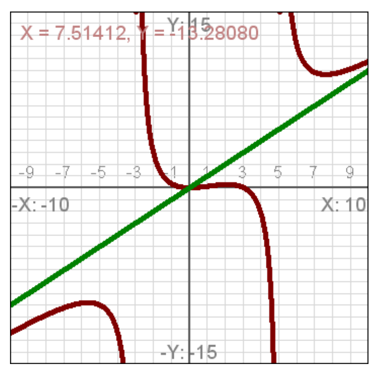
b.



c.



d.



e.