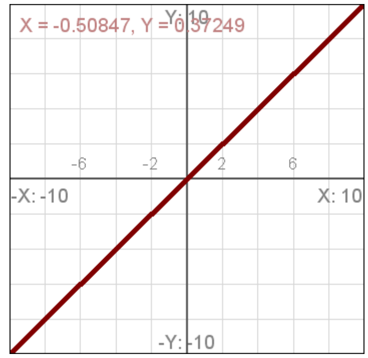
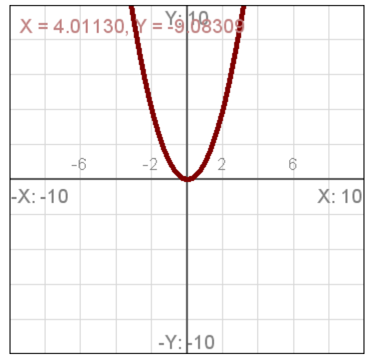
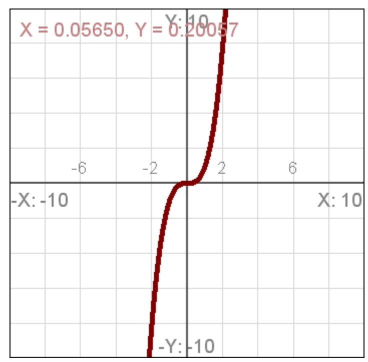
Graphing Real Zeros







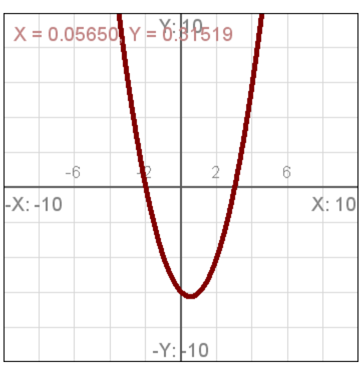
When graphing real zeros of a polynomial, there are basically three types of behaviours that are exemplified by the behaviour of the three simplest polynomials, $y = x, y = x^2, y = x^3$ as shown in the graphs to the right. Polynomials of higher degree can be factored into linear factors with real solutions, or quadratic factors with complex solutions. The complex solutions do not generate *x*-intercepts, and so will not be considered (much) here. The linear factors can be singular or repeated.

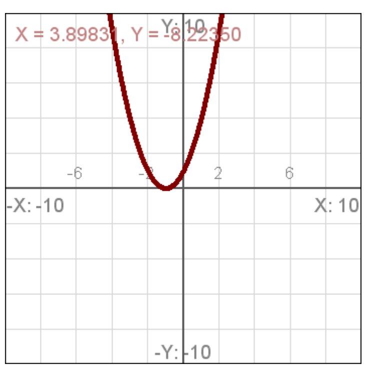
Factors (with degree-1), such as $x, (x-1), (x+3)$, etc. cross the x-axis at the zero as approximately a straight line, like the graph of $y = x$ (shown at top). They could go from positive to negative or negative to positive.

Factors that are repeated can be repeated an even number of times or an odd number of times. Typically, we will encounter degree-2 repeated factors, or occasionally degree-3 repeated factors. Higher degrees are possible, but they are not easily distinguishable from the graphs, so we will be working with the lowest degree that can get the observed behaviour.

The degree-2 repeated factors, such as $x^2, (x-1)^2, (x+3)^2$, touch the *x*-axis at the zero, but remain on the same side of the graph. Like the graph $y = x^2$, it remains positive on both sides of the zero, or remains negative on both sides of the zero. Higher even powers also behave this way.

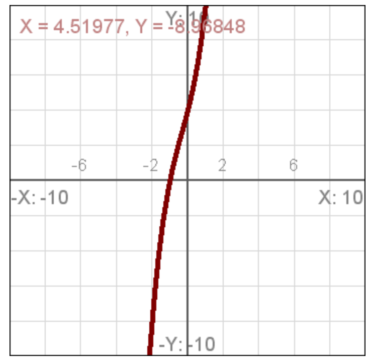
The degree-3 repeated factors, such as $x^3, (x-1)^3, (x+3)^3$, cross the x-axis like other odd degree polynomials, including $y = x, y = x^3$, but degrees higher than 1, like 3, 5, 7, etc. flatten out as they cross the axis, in a way that the linear unrepeated factor does not.



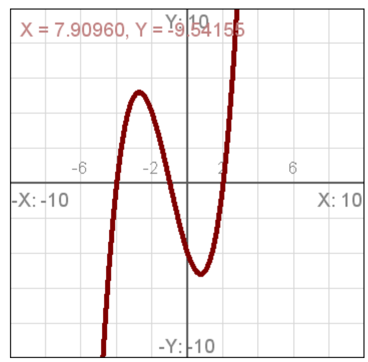


Let's consider some examples of graphs, starting the two possibilities for degree-2 polynomials. (Technically, there is a third case, where there are no zeros, but that doesn't help us illustrate our point.) Consider, then, the case of $f(x) = (x+2)(x-3)$, and $g(x) = (x+1)^2$. The case of $f(x)$, we have two linear factors, so we will get (unrepeated) zeros at -2, and 3. The leading coefficient is 1, so the parabola will open up, and for large |x|, the graph will go to ∞. So starting on the positive side, the graph goes through the axis from positive to negative at -2, and then from negative to positive at 3. In the case of $g(x)$, there is a single repeated zero at -1, but here the graph touches the axis and remains positive on both sides of the zero.

The degree-3 polynomial has more possibilities: i) it could have one real zero and two complex ones; ii) it could have three linear, unrepeated zeros; iii) it could have one linear, and one repeated zero; iv) it could have a degree-3 repeated zero. Examples would be, respectively, $f(x)=(x+1)(x^2+4)$, $g(x)=\frac{1}{2}(x+4)(x+1)(x-2)$, $h(x)=(x+1)^2(x-3)$, and $j(x)=(x-1)^3$. Let's consider the graph of each.

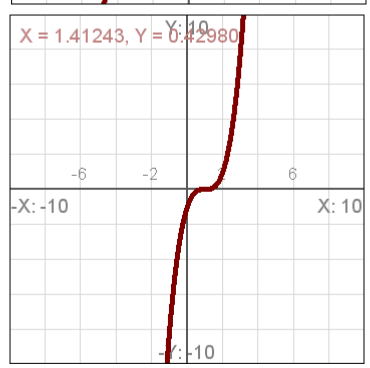


For the graph of $f(x)$, the graph crosses at -1 in essentially a linear fashion. (The curve in the graph above the axis is the result of the complex factor.) Like other odd functions with positive leading coefficients, it starts from -∞ and eventually ends up at ∞ for large |x|.



The graph of $g(x)$ with its three linear factors comes up from -∞ as we would expect, but then have to cross the x-axis three separate times, each time crossing at the zero and changing sign, first at -4, then at -1, and again at 2 before remaining positive thereafter. Each time it crosses the axis, it does so in an essentially linear fashion because the zeros are not repeated.

The graph of $h(x)$ has one linear and one repeated factor. The repeated factor causes the graph to only touch the x-axis at -1 and then turn negative again rather than crossing the x-axis. But at 3, the factor is linear and so the graph crosses in nearly a straight line into positive territory, where it remains since there are no other zeros.



The graph of $j(x)$ is the shifted cubic function with the graph flattening out at the zero before crossing the axis.

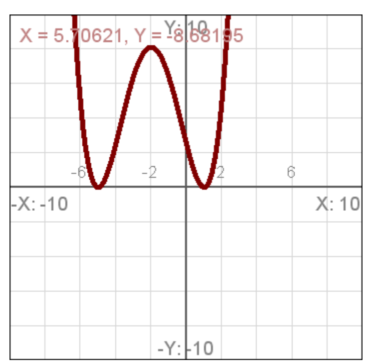
For the quartic graphs below, see if you can match them to the graphs of the functions on the next page (and the last one to the right).

a. $a(x)=\frac{1}{20}(x+5)(x+1)(x-2)(x-4)$

b. $b(x)=\frac{1}{5}(x+1)^2(x-2)(x-5)$

c. $c(x)=\frac{1}{10}(x+5)^2(x-1)^2$

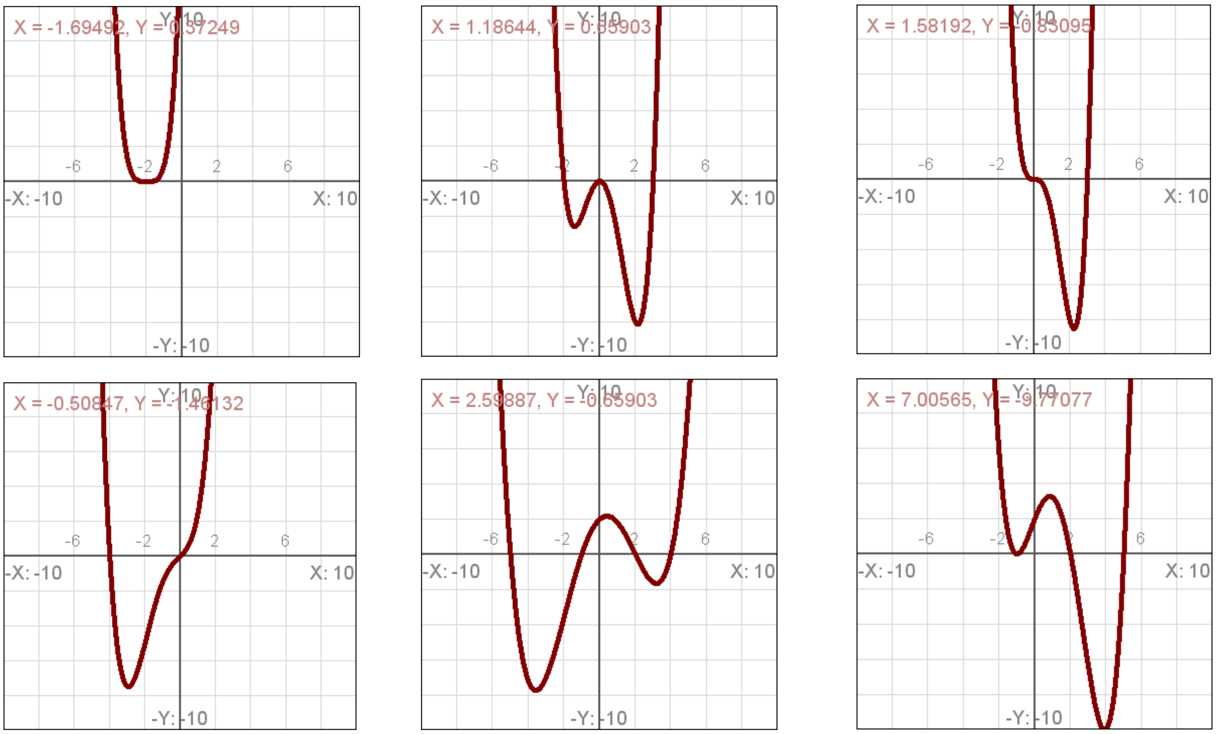
d. $d(x)=x^3(x-3)$



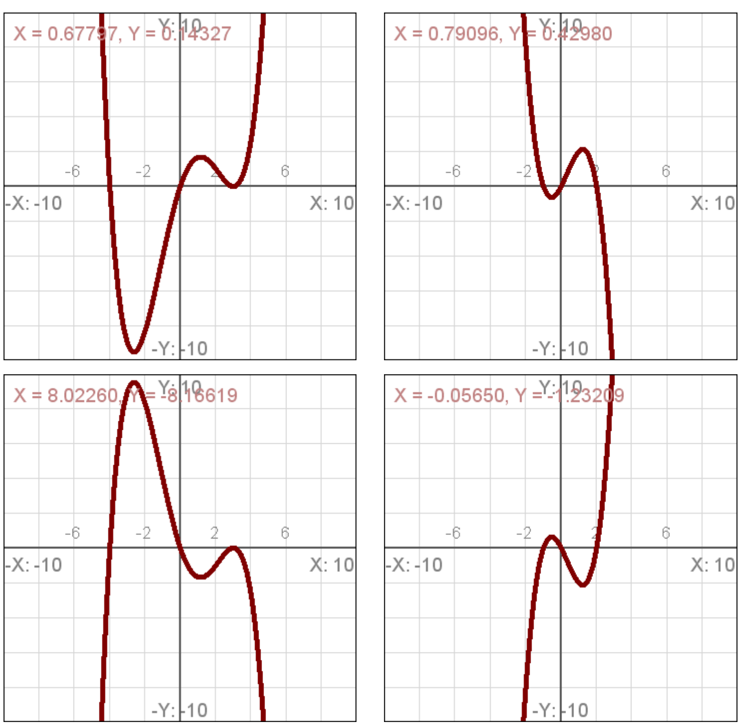
e. $e(x) = \frac{1}{2}x^2(x-3)(x+2)$

f. $f(x) = (x+2)^4$

g. $g(x) = \frac{1}{4}x(x+4)(x^2+1)$



Incidentally, the coefficients are all positive, and don't have an effect on the zeros. They are there only to keep the graphs in the standard window.



What happens when we apply a negative leading coefficient to the graphs? Then the long run behaviour of the graphs flips. For even degree graphs, positive coefficients have the graph going to +∞ for large |x|, but a negative coefficient has it going to -∞: opening up versus opening down. For odd degree graphs, positive coefficients have the graph starting at -∞ for large negative x, and ending up at +∞ for large positive x; however, a negative coefficient starts at +∞ for large negative x, and ends up at -∞ for large positive x. Compare the graphs of

$$f(x) = \frac{1}{12}x(x-3)^2(x+4) \quad \text{vs.}$$

$g(x) = -\frac{1}{12}x(x-3)^2(x+4)$. We can see the same kind of thing with the graphs of $s(x) = x(x+1)(x-2)$ vs. $t(x) = -x(x+1)(x-2)$.

Problems.

h. Sketch the graph of the polynomial $h(x) = (x-1)(x+4)$

i. Sketch the graph of the polynomial $i(x) = -\frac{1}{2}(x-5)(x+3)^2$

j. Sketch the graph of the polynomial $j(x) = \frac{1}{10}(x-2)^2(x+4)(x+5)$

k. Sketch the graph of the polynomial $k(x) = (x-1)^3(x-4)$

l. Sketch the graph of the polynomial $l(x) = -(x-3)(x-2)(x+4)$

m. Sketch the graph of the polynomial $m(x) = x^2(x-1)^3(x+2)$

n. Sketch the graph of the polynomial $n(x) = -\frac{1}{100}(x+2)(x+1)^2(x-2)(x-4)^3$

o. Sketch the graph of the polynomial of degree 4 with zeros at 0, -1, 2 and 3

p. Sketch the graph of the polynomial of degree 2 with zeros at -4, and 5

q. Sketch the graph of the polynomial of degree 3 with zeros at 0 (multiplicity 2), and 3

r. Sketch the graph of the polynomial of degree 5 with zeros at -4, -1(multiplicity 3), and 2

s. State an equation (of lowest degree) that matches each of the graphs below. You may assume that all the zeros are integers.