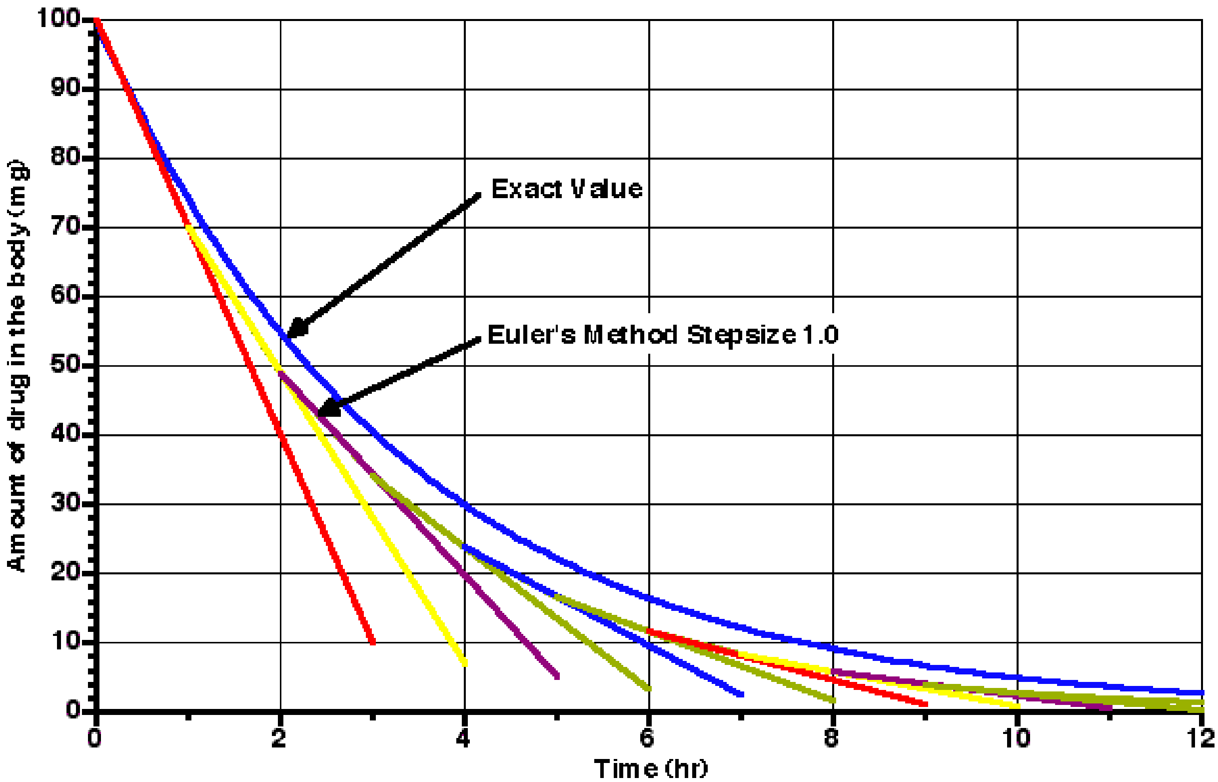
Euler's Method

Euler's method is a method of numerically approximating the solution curve to a differential equation. It is most useful when executed by a computer with a large number of small steps, but we will examine the method primarily with simpler equations and with many fewer steps to get the technique down. Euler's method works on the understanding the that derivative represents the slope of the solution curve at some point, and we make a linear approximation and move along that line until we come to the next step point, and then re-approximate the slope at that point to make the next step. We continue iterating this procedure until we come to our desired stopping point.

Graphically, the method is illustrated below.



As you can see, the method generally tracks the true solution fairly closely, so that you can get an estimate of what is going on over time. Of course, the more complicated the direction field for the differential equation, you will need to take more and smaller steps to remain in the general vicinity of the solution. In general, the larger the steps you take, and the fewer of them there are, the more likely your estimate is to be significantly off of the true value.

Procedurally, we begin with a differential equation, $\frac{dy}{dt} = 2 + t - y$, and an initial point $y_0(1) = 3$. Since the slope of our linear approximation line is the derivative we set $\frac{dy}{dt} = m_1$ at that point.

$$m_1 = 2 + 1 - 3 = 0$$

Then we use this and the initial point (1,3) to come up with the equation of the line we are going to follow.

$$y_1 = m_1 \Delta t + y_0$$

We use the slope we calculated, and the initial starting value for y. Δt is the step size. Generally speaking, this value will be given to you directly or indirectly. A step size may be explicitly stated, such as $\Delta t = 0.1$, or a value you wish to approximate, say y at time t and the number of steps you are to use (n). In which case you can calculate the step size by $\Delta t = \frac{t - t_0}{n}$.

So, suppose we wanted to approximate the solution to the system at t=2, and we're going to calculate 2 steps. So then $\Delta t = \frac{2-1}{2} = 0.5$ (Since this is an approximation, expect to use decimals, and for most problems, I recommend rounding to at least two digits more than you want in the final solution.)

$$y_1 = 0(0.5) + 3 = 3$$

Since the slope was zero at the start, it didn't move anywhere in the y-direction. Now we increment our time by Δt: $t_1 = 1 + \Delta t = 1 + 0.5 = 1.5$, and then recalculate the slope at the new point, and proceed as before.

$$m_2 = 2 + 1.5 - 3 = 0.5$$
$$y_2 = 0.5(0.5) + 3 = 3.25$$
$$t_2 = 1.5 + \Delta t = 1.5 + 0.5 = 2$$

We stop when we get to the target time, and report the y-value. Thus $y(t) \approx 3.25$.

Excel is extremely useful here if we have many steps.

If we wanted to approximate the same 1 time-unit change, but this time in ten steps instead of two ($\Delta t = 0.1$), we can set up these formulas easily to auto-calculate.

Step	Y begin	T begin	slope	Y stop	T new
1	3	1	0	3	1.1
2	3	1.1	0.1	3.01	1.2
3	3.01	1.2	0.19	3.029	1.3
4	3.029	1.3	0.271	3.0561	1.4
5	3.0561	1.4	0.3439	3.09049	1.5
6	3.09049	1.5	0.40951	3.131441	1.6
7	3.131441	1.6	0.468559	3.178297	1.7
8	3.178297	1.7	0.521703	3.230467	1.8
9	3.230467	1.8	0.569533	3.28742	1.9
10	3.28742	1.9	0.61258	3.348678	2

We get a slightly different prediction, but in the same general vicinity.

If we complete the same calculation for 100 steps ($\Delta t = 0.01$), we get an estimate of about 3.366.

This differential equation can be solved explicitly as a linear first order differential equation.

Practice Problems.

Use Euler's method to approximate the solution at the specified point using the requested number of steps, or the given step size. You should do at least three steps by hand (for any problem with 3 or more steps), but the remainder you can compute in Excel or another similar program.

1. $\frac{dy}{dt} = 2y - 1, y_0(1) = 0, \Delta t = 0.5, y(2) = ?$
2. $\frac{dy}{dt} = \cos(t) - 2y, y_0(0) = 4, \Delta t = 1, y(2) = ?$
3. $\frac{dy}{dt} = 5 - 3\sqrt{y}, y_0(1) = 4, \Delta t = 0.1, y(1.5) = ?$
4. $\frac{dy}{dt} = y(2 - ty), y_0(2) = 1, n = 3, y(3) = ?$
5. $\frac{dy}{dt} = t^2 + y^2, y_0(1) = 0, n = 5, y(3) = ?$
6. $\frac{dy}{dt} = -ty + 1, y_0(0) = 1, n = 10, y(1) = ?$
7. $\frac{dy}{dt} = \frac{3t^2}{y^2-4}, y_0(3) = 1, n = 20, y(7) = ?$