Reduction of Order

A useful technique in solving differential equations is reduction of order. For a variety of reasons, we may be able to find one solution to a differential equation before the others, either by guessing, or using some advanced techniques like power series solutions. Reduction of order is a technique whereby we use the proposed solution to reduce our differential equation one order, in the hope that the reduced equation will be easier to solve. This technique can potentially apply to any differential equation higher than first order. In the examples below, we will illustrate it primarily with second and third order equations.

There are some useful formulas from calculus that we can derive that will aid our procedures greatly in this process.

Recall the product rule for derivatives: $(fg)' = f'g + fg'$. Using this as a basis we can derive a product rule for derivatives of three products, for $(fgh)'$.

$$(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$$

In addition, we may also be interested in finding second or third derivatives of product rules. We will derive both of these formulas below.

$$(fg)'' = [(fg)']' = [f'g + fg']' = f''g + f'g' + f'g' + fg'' = f''g + 2f'g' + fg''$$

This looks a lot like a perfect square trinomial, so the third derivative will also look like a binomial cubed.

$$\begin{aligned}(fg)''' = [(fg)'']' = [f''g + 2f'g' + fg'']' &= f'''g + f''g' + 2f'' + g' + 2f'g'' + f'g'' + fg''' \\ &= f'''g + 3f''g' + 3f'g'' + fg'''\end{aligned}$$

Since our technique will involve multiplying the given solution by a dummy solution that we hope to find, these situations are bound to arise.

Let us do one more formula, just in case it's needed, the second derivative for a triple product.

$$\begin{aligned}(fgh)'' &= [(fgh)']' = [f'gh + fg'h + fgh']' \\ &= f''gh + f'g'h + f'gh' + f'g'h + fg''h + fg'h' + f'gh' + fg'h' + fgh'' \\ &= f''gh + 2f'g'h + 2f'gh' + fg''h + 2fg'h' + fgh''\end{aligned}$$

You can tell from the formulas that this process can be a bit tedious, but it is effective, and it should be less tedious now that we've derived these formulas.

The general procedure for reduction of order is to take a known solution to the system $y_1(t)$ and propose a second solution to the equation is of the form $y_1(t)v(t)$. When we plug this into the differential equation, we will get a system in v(t) and its derivatives only, an equation that will hopefully be simpler to solve. Because the form of the solution is a product, we will use the above formulas to take our derivatives. The triple product case arises when $y_1(t)$ is itself a product. This is uncommon, but it does occur.

Example 1.
Use reduction of order to find the second solution for $t^2y'' - 4ty' + 6y = 0, t > 0, y_1(t) = t^2$.

We assume the solution is of the form $y_2(t) = y_1(t)v(t) = t^2v(t)$. We will need both the first and second derivatives for this function. For that, we will use the formulas we derived above.

$$y_2' = 2tv + t^2v'$$
$$y_2'' = 2v + 2(2t)v' + t^2v'' = 2v + 4tv' + t^2v''$$

Next, we plug these expressions into the equation and see what cancels out. Typically, all the $v(t)$ terms cancel, but sometimes the $v'(t)$ also cancel.

$$t^2[2v + 4tv' + t^2v''] - 4t[2tv + t^2v'] + 6[t^2v] = 0$$
$$2t^2v + 4t^3v' + t^4v'' - 8t^2v - 4t^3v' + 6t^2v = 0$$
$$t^4v'' = 0$$

Since both the v and v' terms cancelled, we can divide by the multiplier of v'', and end up with just $v'' = 0$.

To solve this, just integrate.

$$v' = \int v''\,dt = \int 0\,dt = C$$
$$v = \int v'\,dt = \int C\,dt = Ct$$

For this procedure we can ignore the constant of integration on the last step since the function is going to be multiplied by our original solution, and so will only produce another multiple of y_1. In fact, in general, the constants will also be taken care of by the general constants we use as solutions in $y(t) = c_1y_1(t) + c_2y_2(t)$.

To interpret this result, we are interested in the function of t, which in this case, gives us $v(t) = t$, and so $y_2(t) = t^2v(t) = t^2 \cdot t = t^3$.

Our general solution to the system is therefore $y(t) = c_1t^2 + c_2t^3$. You can check this in the original equation by taking derives and verifying that the equation gives you 0. Once we learn how to solve Cauchy-Euler equations, of which this is one, we'll be able to verify the solution by another means.

Example 2.
Use reduction of order to find the second solution for $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, x > 0, y_1(x) = \frac{1}{\sqrt{x}}\sin(x)$.

We assume $y_2(x) = \frac{1}{\sqrt{x}}\sin(x)\,v(x) = x^{-\frac{1}{2}}\sin(x)\,v(x)$. For this problem, we'll get to use our triple product formulas.

$$y_2' = f'gh + fg'h + fgh' = -\frac{1}{2}x^{-\frac{3}{2}}\sin(x)\,v + x^{-\frac{1}{2}}\cos(x)\,v + x^{-\frac{1}{2}}\sin(x)\,v'$$

$$\begin{aligned} y_2'' &= f''gh + 2f'g'h + 2f'gh' + fg''h + 2fg'h' + fgh'' \\ &= \frac{3}{4}x^{-\frac{5}{2}}\sin(x)\,v + 2\left(-\frac{1}{2}x^{-\frac{3}{2}}\cos(x)\,v\right) + 2\left(-\frac{1}{2}x^{-\frac{3}{2}}\sin(x)\,v'\right) + x^{-\frac{1}{2}}(-\sin(x))v \\ &\quad + 2x^{-\frac{1}{2}}\cos(x)\,v' + x^{-\frac{1}{2}}\sin(x)\,v'' \\ &= \frac{3}{4}x^{-\frac{5}{2}}\sin(x)\,v - x^{-\frac{3}{2}}\cos(x)\,v - x^{-\frac{3}{2}}\sin(x)\,v' - x^{-\frac{1}{2}}\sin(x)\,v + 2x^{-\frac{1}{2}}\cos(x)\,v' \\ &\quad + x^{-\frac{1}{2}}\sin(x)\,v'' \end{aligned}$$

Put these into the original equation.

$$\begin{aligned} &x^2\left[\frac{3}{4}x^{-\frac{5}{2}}\sin(x)\,v - x^{-\frac{3}{2}}\cos(x)\,v - x^{-\frac{3}{2}}\sin(x)\,v' - x^{-\frac{1}{2}}\sin(x)\,v + 2x^{-\frac{1}{2}}\cos(x)\,v' + x^{-\frac{1}{2}}\sin(x)\,v''\right] \\ &\quad + x\left[-\frac{1}{2}x^{-\frac{3}{2}}\sin(x)\,v + x^{-\frac{1}{2}}\cos(x)\,v + x^{-\frac{1}{2}}\sin(x)\,v'\right] + \left(x^2 - \frac{1}{4}\right)x^{-\frac{1}{2}}\sin(x)\,v = 0 \end{aligned}$$

$$\begin{aligned} &\frac{3}{4}x^{-\frac{1}{2}}\sin(x)\,v - x^{\frac{1}{2}}\cos(x)\,v - x^{\frac{1}{2}}\sin(x)\,v' - x^{\frac{3}{2}}\sin(x)\,v + 2x^{\frac{3}{2}}\cos(x)\,v' + x^{\frac{3}{2}}\sin(x)\,v'' \\ &\quad - \frac{1}{2}x^{-\frac{1}{2}}\sin(x)\,v + x^{\frac{1}{2}}\cos(x)\,v + x^{\frac{1}{2}}\sin(x)\,v' + x^{\frac{3}{2}}\sin(x)\,v - \frac{1}{4}x^{-\frac{1}{2}}\sin(x)\,v = 0 \\ &\left(\frac{3}{4}x^{-\frac{1}{2}}\sin(x)\,v - \frac{1}{2}x^{-\frac{1}{2}}\sin(x)\,v - \frac{1}{4}x^{-\frac{1}{2}}\sin(x)\,v\right) + \left(-x^{\frac{1}{2}}\cos(x)\,v + x^{\frac{1}{2}}\cos(x)\,v\right) \\ &\quad + \left(-x^{\frac{3}{2}}\sin(x)\,v + x^{\frac{3}{2}}\sin(x)\,v\right) + \left(-x^{\frac{1}{2}}\sin(x)\,v' + x^{\frac{1}{2}}\sin(x)\,v'\right) + 2x^{\frac{3}{2}}\cos(x)\,v' \\ &\quad + x^{\frac{3}{2}}\sin(x)\,v'' = 0 \end{aligned}$$

All the terms in parentheses in the last line cancel, leaving just the last two terms.

$$2x^{\frac{3}{2}}\cos(x)\,v' + x^{\frac{3}{2}}\sin(x)\,v'' = 0$$

Divide out the $x^{\frac{3}{2}}$.

$$2\cos(x)\,v' + \sin(x)\,v'' = 0$$

We can treat this equation as a first order equation if we allow $v' = u$.

$$2\cos(x)\,u + \sin(x)\,u' = 0$$

This equation can be solved by separation of variables.

$$\sin(x)\frac{du}{dx} = -2\cos(x)\,u$$

$$\frac{du}{u} = -2\cot(x)\,dx$$
$$\int \frac{du}{u} = \int = -2\cot(x)\,dx$$
$$\ln|u| = -2\ln|\sin(x)| + C$$
$$\ln|u| = \ln|csc^2(x)| + C = \ln|Acsc^2(x)|$$
$$v' = u = Acsc^2(x)$$

And so we integrate again to obtain $v(x)$. We again ignore the constant as we discussed in Example 1.

$$v = \int v'\,dx = \int csc^2(x)\,dx = -\cot(x)$$

Therefore, $y_2(x) = \frac{1}{\sqrt{x}}\sin(x)\cot(x) = \frac{1}{\sqrt{x}}\cos(x)$. This means the general solution to the system is $y(x) = c_1\frac{1}{\sqrt{x}}\sin(x) + c_2\frac{1}{\sqrt{x}}\cos(x)$.

Example 3.
Use reduction of order to reduce the third order equation to a second order, and find the remaining two solutions for the equation $(2-t)y''' + (2t-3)y'' - ty' + y = 0, t < 2, y_1(t) = e^t$.

Assume $y_2(t) = e^t v(t)$. We will need three derivatives.

$$y_2' = e^t v + e^t v'$$
$$y_2'' = e^t v + 2e^t v' + e^t v''$$
$$y_2''' = e^t v + 3e^t v' + 3e^t v'' + e^t v'''$$

Plug this into the original equation and reduce.

$$(2-t)(e^t v + 3e^t v' + 3e^t v'' + e^t v''') + (2t-3)(\,e^t v + 2e^t v' + e^t v'') - t(e^t v + e^t v') + e^t v = 0$$
$$2e^t v + 6e^t v' + 6e^t v'' + 2e^t v''' - te^t v - 3te^t v' - 3te^t v'' - te^t v''' + 2te^t v + 4te^t v' + 2te^t v''$$
$$- 3e^t v - 6e^t v' - 3e^t v'' - te^t v - te^t v' + e^t v = 0$$

$$(2e^t v''' - te^t v''') + (6e^t v'' - 3te^t v'' + 2te^t v'' - 3e^t v'')$$
$$+ (6e^t v' - 3te^t v' + 4te^t v' - 6e^t v' - te^t v')$$
$$+ (2e^t v - te^t v + 2te^t v - 3e^t v - te^t v + e^t v) = 0$$

$$(2-t)e^t v''' + (3-t)e^t v'' = 0$$

Since the last two parentheses cancel completely. Divide out the e^t, and make a substitution for $v'' = u$.

$$(2-t)u' + (3-t)u = 0$$
$$(2-t)\frac{du}{dt} + (3-t)u = 0$$

We can solve this equation by separation of variables.

$$(2-t)\frac{du}{dt} = -(3-t)u$$
$$\frac{du}{u} = -\frac{t-3}{t-2}dt$$
$$\int \frac{du}{u} = -\int \frac{t-3}{t-2}dt$$

To integrate the right side of the equation, we will need to do long division first.

$$\int \frac{du}{u} = -\int 1 - \frac{1}{t-2}dt$$
$$\ln|u| = -t + \ln|t-2| + C = \ln|Ae^{-t}(t-2)|$$
$$u = e^{-t}(t-2)$$

Recall that this is equal to v'', so we will need to integrate this (by parts) twice more to obtain v.

$$v' = \int v'' = \int e^{-t}(t-2)\,dt = (t-2)e^{t} - e^{t} = (t-3)e^{t}$$
$$v = \int v' = \int (t-3)e^{t}dt = (t-3)e^{t} - e^{t} = (t-4)e^{t}$$

This gives us $y_2(t) = y_1(t)v(t) = e^{t}(t-4)e^{t} = (t-4)e^{2t}$

This actually gives us all three solutions for the equation (since degree three equations should have three solutions), since te^{2t} is one solution, and e^{2t} is the second. Again, we can ignore the constant multipliers since they will be accounted for in the unknown constants in our general equation.

$$y(t) = c_1e^{t} + c_2te^{2t} + c_3e^{2t}$$

Practice Problems.
Use reduction of order and the given solution to find the remaining solution(s) of the given equations.

1. $y'' - y' - 2y = 0, y_1(t) = e^{-t}$
2. $(x-1)y'' - xy' + y = 0, x > 0, y_1(x) = e^{x}$
3. $t^2y'' + 2ty' - 2y = 0, t > 0, y_1(t) = t$
4. $xy'' - y' + 4x^3y = 0, x > 0, y_1(x) = \sin(x^2)$
5. $2x^2y'' - xy' + y = 0, x > 0, y_1(x) = \sqrt{x}$
6. $(x^2+1)y'' - 2xy' + 2y = 0, y_1(x) = x$
7. $y'' - \tan(x)\,y' + 2y = 0, -\frac{\pi}{2} < x < \frac{\pi}{2}, y_1(x) = \sin(x)$
8. $t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, t > 0, y_1(t) = t^2, y_2(t) = t^3$ [Hint: do this in two steps.]