Differential Equations Concentration Problems Key

1. $R_{in} = \frac{5 \text{ ppm}}{\text{gal}} \cdot \frac{5 \text{ gal}}{\text{min}} = 25 \frac{\text{ppm}}{\text{min}}$

$R_{out} = \frac{A}{40{,}000} \cdot \frac{5 \text{gal}}{\text{min}} = \frac{A}{8000} \frac{\text{ppm}}{\text{min}}$

$\frac{dA}{dt} = 25 - \frac{A}{8000} = -\frac{1}{8000}(A - 200{,}000)$ $\quad \int \frac{dA}{A-200{,}000} = \int -\frac{1}{8000}\, dt$

$\ln|A - 200{,}000| = -\frac{1}{8000}t + C \Rightarrow A - 200{,}000 = A_0 e^{-\frac{1}{8000}t}$

$A(t) = 200{,}000 + A_0 e^{-\frac{1}{8000}t}$ $\quad A(0) = 0$

$0 = 200{,}000 + A_0 \cancel{e^0}$

$\Rightarrow A_0 = -200{,}000$

$A(t) = 200{,}000 - 200{,}000 e^{-\frac{1}{8000}t}$

$15 = 200{,}000 - 200{,}000 e^{-\frac{1}{8000}t} \Rightarrow \frac{-199{,}985}{-200{,}000} = e^{-\frac{1}{8000}t}$

$\ln\left|\frac{199{,}985}{200{,}000}\right| \cdot (-8000) = t \approx .6$ minutes

≈ 36 seconds

2. $A(0) = 5$ lbs.

$R_{in} = \frac{8 \text{ gal}}{\text{hr}} \cdot \frac{\sin t \text{ lbs}}{\text{gal}} = 8\sin(t) \frac{\text{lbs}}{\text{hr}}$

rate in for water = 8
rate out " " = 6
8 − 6 = 2

$R_{out} = \frac{A}{1200+2t} \cdot \frac{6 \text{ gal}}{\text{hr}} = \frac{6A}{1200+2t} \frac{\text{lbs}}{\text{hr}}.$

$\frac{dA}{dt} = 8\sin t - \frac{\cancel{6}^{3} A}{\cancel{1200+2t}_{600+t}} \Rightarrow A' + \frac{3}{600+t} A = 8\sin t$

$\mu = e^{\int \frac{3}{600+t} dt} = e^{3\ln|600+t|} = (600+t)^3$

$(600+t)^3 A' + 3(600+t)^2 A = 8\sin t \cdot (600+t)^3$

$\int \left[(600+t)^3 A\right]' = \int 8\sin t \cdot (600+t)^3\, dt$

2 cont'd.

$$(600+t)^3 A =$$

$$8(600+t)^3 \cos t + 24(600+t)^2 \sin t$$
$$- 48(600+t)\cos t - 48 \sin t + C$$

+/-	u	dv
+	$(600+t)^3$	$8 \sin t$
−	$3(600+t)^2$	$8 \cos t$
+	$6(600+t)$	$-8 \sin t$
−	6	$-8 \cos t$
+	0	$8 \sin t$

$$A = 8\cos t + \frac{24 \sin t}{600+t} - \frac{48 \cos t}{(600+t)^2}$$

$$- \frac{48 \sin t + C}{(600+t)^3}$$

$A(0)=5$

$$5 = 8 + 0 - \frac{48}{600^2} - \frac{C}{(600)^3} \Rightarrow +3 = \frac{48}{600^2} + \frac{C}{600^3}$$

$$6.48 \times 10^8 = 28{,}800 + C$$
$$C = 647{,}971{,}200$$

$$A(t) = 8\cos t + \frac{24 \sin t}{600+t} - \frac{48 \cos t}{(600+t)^2} - \frac{48 \sin t + 647{,}971{,}200}{(600+t)^3}$$

1500 gal $\Rightarrow$ tank overflows after 300 gallons added since 2 gallons added per hours: 150 hrs. = t

$$A(150) = 8\cos(150) + \frac{24 \sin(150)}{750} - \frac{48 \cos(150)}{750^2} - \frac{48 \sin(150) + 647{,}971{,}200}{750^3}$$

$A(150) = 4.035$ lbs. (be sure you are in radian mode)

This is a clever problem, but $\sin(t)$ lbs/gal isn't realistic since you can add "negative lbs of salt" as you would half the time according to $\sin(t)$.

3. $A(0)=25$

$$R_{in} = \frac{1.5 \text{ lbs}}{\cancel{\text{gal}}} \cdot \frac{2 \cancel{\text{gal}}}{\text{min}} = \frac{3 \text{ lbs}}{\text{min}}$$

$$R_{out} = \frac{A}{140+t} \cdot \frac{1 \cancel{\text{gal}}}{\text{min}} = \frac{A}{140+t}$$

diff between rate in & out for water is $2-1=1$

$$\frac{dA}{dt} = 3 - \frac{A}{140+t} \Rightarrow A' + \frac{1}{140+t} A = 3$$

$$\mu = e^{\int \frac{1}{140+t} dt} = e^{\ln|140+t|} = 140+t$$

$$(140+t)A' + A = 420 + 3t$$

$$\int [(140+t)A]' = \int 420 + 3t \, dt \qquad (\cancel{140+t})A = \frac{420t + \frac{3}{2}t^2 + C}{140+t}$$

$$25 = \frac{0+0+C}{140} \Rightarrow C = 3500$$

$$A(t) = \frac{420t + \frac{3}{2}t^2 + 3500}{140+t}$$

$$A(60) = \frac{420(60) + \frac{3}{2}(60)^2 + 3500}{200}$$

1 hr = 60 min
t in minutes

$$A(60) = 170.5 \text{ lbs}.$$

4. $R_{in} = \frac{20 \cancel{L}}{\text{min}} \cdot \frac{0 \text{ kg}}{\cancel{L}}$ $\quad R_{out} = \frac{A \text{ kg}}{100 \cancel{L}} \cdot \frac{20 \cancel{L}}{\text{min}} = \frac{A}{5}$

$$\frac{dA}{dt} = 0 - \frac{A}{5} \Rightarrow \int \frac{dA}{A} = \int -\frac{1}{5} dt \Rightarrow \ln A = -\frac{1}{5}t + C$$

$$A = A_0 e^{-\frac{1}{5}t}$$

$A(0) = 10$ kg $\quad A_0 = 10$

$$\boxed{A(t) = 10 e^{-\frac{1}{5}t}}$$