Hyperbolic Trig Functions

Hyperbolic trig functions take their name from the many properties they share with regular (circular) trig functions. The two primary functions hyperbolic sine (sinh) and hyperbolic cosine (cosh) are defined as follows:

$$\sinh(t) = \frac{e^t - e^{-t}}{2}$$
$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

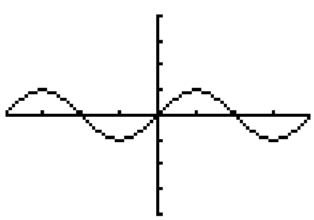
Compare these definition with the identities for sine and cosine derived from Euler's formula: $e^{ix} = \cos(x) + isin(x)$.

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$
$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

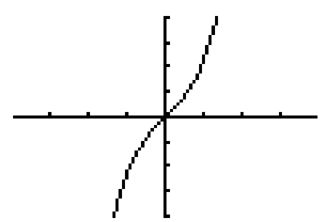
Because of their similarities, hyperbolic trig functions have many similar properties to the regular trig functions. For instance, sine and hyperbolic sine are both odd functions, with sin(0)=0 and sinh(0)=0. Likewise, cosine and hyperbolic cosine are both even functions, with cos(0)=1 and cosh(0)=1.

We can see these properties most easily from the symmetry of their graphs.

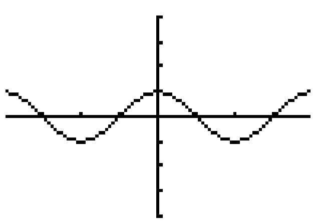
Sine:



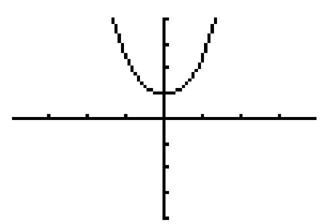
Hyperbolic Sine:



Cosine:



Hyperbolic Cosine:



While they do share the same symmetry, hyperbolic trig functions are not periodic. They are defined for all values of x.

The other four hyperbolic trig functions are defined the same way as the other four regular trig functions are in terms of sine and cosine.

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \tanh(x) = \frac{\sinh(x)}{\cos\text{h}(x)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \coth(x) = \frac{\cos\text{h}(x)}{\sinh(x)} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$$

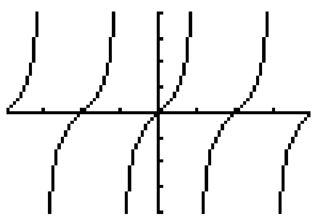
$$\sec(x) = \frac{1}{\cos(x)} \qquad \text{sech}(x) = \frac{1}{\cos\text{h}(x)} = \frac{2}{e^t + e^{-t}}$$

$$\csc(x) = \frac{1}{\sin(x)} \qquad \text{csch}(x) = \frac{1}{\sin\text{h}(x)} = \frac{2}{e^t - e^{-t}}$$

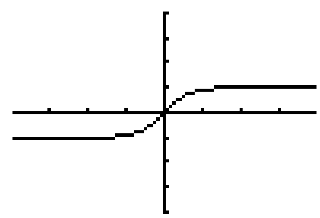
It takes a long time to say "hyperbolic sine" and "hyperbolic cosine", and so forth all the time. So mathematicians have shorthands for all these terms based on their abbreviations. Hyperbolic sine is called "sinch", hyperbolic cosine "kosh", hyperbolic tangent "tanch", hyperbolic cotangent "cotanch", hyperbolic secant "seech", and hyperbolic cosecant "coseech".

Let's compare the graphs of each of these functions.

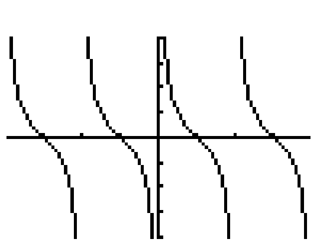
Tangent:



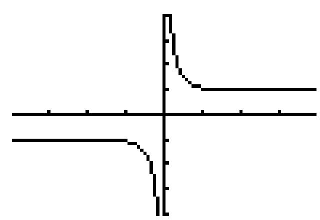
Hyperbolic Tangent:



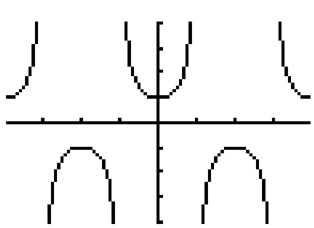
Cotangent:



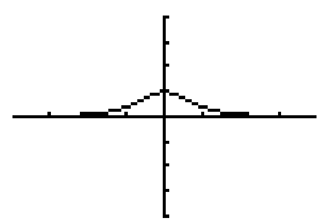
Hyperbolic Cotangent:

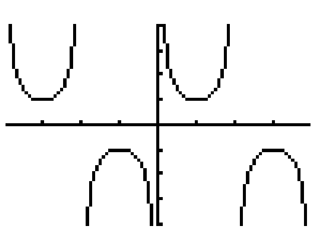
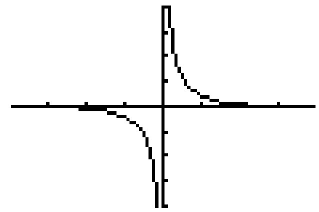


Secant:



Hyperbolic Secant:



Cosecant:  Hyperbolic Cosecant: 

All the hyperbolic trig graphs maintain the same symmetry as their regular trig counterparts, and we can find their values at 0 (if they are defined there) by using what we know about the hyperbolic sine and hyperbolic cosine graphs at zero.

More similarities with regular trig functions abound, including in some of their identity relationships (differing from regular trig identities by the occasional minus sign), and in their derivatives (also occasionally differing by negative signs).

Table of identities:

$cos^2(x) + sin^2(x) = 1$	$cosh^2(x) - sinh^2(x) = 1$
$1 + tan^2(x) = sec^2(x)$	$1 = tanh^2(x) + sech^2(x$
$1 + cot^2(x) = csc^2(x)$	$1 = coth^2(x) - csch^2(x)$
$sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$	$sinh^2(x) = -\frac{1}{2}[1 - \cos h(2x)]$
$cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$	$cosh^2(x) = \frac{1}{2}[1 + \cos h(2x)]$
$\sin(2x) = 2\sin(x)\cos(x)$	$\sin h(2x) = 2\sin h(x)\cosh(x)$
$\cos(2x) = cos^2(x) - sin^2(x)$	$\cosh(2x) = cosh^2(x) + sinh^2(x)$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$	$\sin h(x \pm y) = \sin h(x)\cos \mathrm{h}(y) \pm \cos h(x)\sinh(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$	$\cosh(x \pm y) = \cos h(x)\cos \mathrm{h}(y) \pm \sinh(x)\sinh(y)$

Table of derivatives:

$\frac{d}{dx}[\sin(x)] = \cos(x)$	$\frac{d}{dx}[\sinh(x)] = \cosh(x)$
$\frac{d}{dx}[\cos(x)] = -\sin(x)$	$\frac{d}{dx}[\cosh(x)] = \sinh(x)$
$\frac{d}{dx}[\tan(x)] = sec^2(x)$	$\frac{d}{dx}[\tanh(x)] = sech^2(x)$
$\frac{d}{dx}[\cot(x)] = -csc^2(x)$	$\frac{d}{dx}[\coth(x)] = -csch^2(x)$
$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$	$\frac{d}{dx}[\mathrm{sech}(x)] = -\sec \mathrm{h}(x)\tanh(x)$
$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	$\frac{d}{dx}[\mathrm{csch}(x)] = -\csc \mathrm{h}(x)\coth(x)$

Hyperbolic trig functions are most common used in differential equations as an alternate means of dealing with exponential functions, in part because of their symmetry properties (which exponentials lack) and because of their similarities to the standard trig functions. They also have Taylor series similar to those of the regular trig functions and so appear frequently in series solutions.

Taylor series identities:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \sin h(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \cos h(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Example 1. Prove that the derivative of sinh(x) is cosh(x) using the definition of the hyperbolic trig functions.

$$\frac{d}{dx}[\sinh(x)] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}[e^x - e^{-x}] = \frac{1}{2}[e^x + e^{-x}] = \cosh(x)$$

Example 2. Use the Taylor series of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to prove that the Taylor series of sinh(x)= $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ when the series is centered at 0.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots$$

Subtract these expressions:

$$e^x - e^{-x} = 2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + 2\frac{x^7}{7!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

And then re-index for odd numbers only:

$$\sin h(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Example 3. Show that for the differential equation $y'' - y = 0$, the set of solutions $y_1 = e^t, y_2 = e^{-t}$ and $y_1 = \cosh(\mathrm{t}), y_2 = \sinh(t)$ both satisfy the differential equation, and both form a fundamental set of solutions.

For set #1: $y'_1 = e^t, y'_2 = -e^{-t}$ and $y''_1 = e^t, y''_2 = e^{-t}$

$$Ae^{t}+Be^{-t}-(Ae^{t}+Be^{-t})=0$$

$$W=\begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix}=-1-1=-2\neq 0$$

For set #2: $y'_1=\sinh(\mathrm{t})\,, y'_2=\cosh(t)$ and $y''_1=\cosh(\mathrm{t})\,, y''_2=\sinh(t)$

$$Acosh(t)+Bsinh(t)-\big(Acosh(t)+Bsinh(t)\big)=0$$

$$W=\begin{vmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{vmatrix}=cosh^2(t)-sinh^2(t)=1\neq 0$$

Practice Problems:

1. Prove that $\frac{d}{dx}[\tanh(x)]=sech^2(x)$ using the definitions of hyperbolic tangent and hyperbolic secant. Do it with the exponential definitions, and with the sinh(x) and cosh(x) definitions.
2. Use the exponential definitions of sinh(x) and cosh(x) to prove the identity $cosh^2(x)-sinh^2(x)=1$.
3. Use the Taylor series of the exponential function provided in example 2 to show that the Taylor series for cosh(x) can be derived as I did for sinh(x).
4. Give integration formulas for sinh(x), cosh(x), tanh(x) and coth(x). For the latter two, use substation and the integrals for sinh(x) and cosh(x). Can you find a similar formula for sech(x) and csch(x)? [Hint: use the formulas for the regular trig functions and work backwards. Replace the regular trig functions with their hyperbolic counterparts and take the derivative. Do you need to change signs anywhere?]
5. Solve the following differential equations using both exponentials and hyperbolic trig functions and show that both forms of the solution are fundamental sets.
   a. $y''-4y=0$
   b. $y''-9y=0$
6. Use the Taylor series formulas for sinh(x) and cosh(x) to prove the derivative formulas for sinh(x) and cosh(x).

Final Note: These functions are available in your calculator. In the TI-84, you can find them in the Catalog 

. Then scroll through the list to find them.