Laplace Transforms

Laplace transforms show up in a number of different applications including engineering and in quantum mechanics. It gives us a way of transforming functions (particularly differential equations) in the t-domain into a function (algebraic equation) in the s-domain. We have many more tools available to us in algebra than in differential equations, and so this can be useful for us to solve differential equations we might otherwise have more difficulty with. This handout will focus mainly on performing the transform formulas, and some basic skills, as well as using a table of transform formulas to find the inverse Laplace transform. We will solve a simple differential equation at the end to illustrate the procedure.

1. Laplace Transform definition

The Laplace transform for any general function f(t) is given by $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$.

We will use the notation $\mathcal{L}\{f(t)\} = F(s)$ throughout to indicate a Laplace transform. All of the formulas that exist for Laplace transforms that you find in tables are obtained from applying this definition. Let's consider two examples to illustrate.

Example 1. Find $\mathcal{L}\{1\}$.

To find the Laplace transform of {1}, we'll integrate $\int_0^\infty e^{-st} \cdot 1dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$

We can see from the properties of integrals that if we find the Laplace transform of any constant c, we have $\mathcal{L}\{c\} = \int_0^\infty e^{-st} cdt = c\int_0^\infty e^{-st} dt = c\left(\frac{1}{s}\right)$.

This property of constants can be shown for any function f(t), since when we integrate cf(t), we'll always be able to pull this constant out of the integral. Thus $\mathcal{L}\{cf(t)\} = cF(s)$.

Because Laplace transforms are based on an integration formula, other properties of integrals will also hold, such as integrating term-by-term.

Example 2. Find $\mathcal{L}\{2t + 1\}$.

To find this, integrate $\int_0^\infty e^{-st}(2t + 1)dt = 2\int_0^\infty te^{-st} dt + \int_0^\infty e^{-st} dt$. The second term here is just $\mathcal{L}\{1\}$, which we already found. The second term is $2\mathcal{L}\{t\}$. To find this, we'll need to integrate by parts.

Let u=t, and $dv = e^{-st}$. This gives us du=dt, and $v = \frac{e^{-st}}{-s}$. So we get $\int_0^\infty te^{-st} dt = \frac{te^{-st}}{-s} - \int_0^\infty \frac{e^{-st}}{-s} dt = \left.\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_0^\infty = 0 + 0 + 0 - \frac{-1}{s^2} = \frac{1}{s^2}$.

Thus $\mathcal{L}\{2t + 1\} = \frac{2}{s^2} + \frac{1}{s}$.

Example 3. Find $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 2t+1, & 0 \le t \le 1 \\ 3, & t \ge 1 \end{cases}$

We'll have a shortcut way to handle piecewise functions later, but for now, we can do it by the definition.

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt &= \int_0^1 e^{-st}(2t+1)dt + \int_1^\infty 3e^{-st}dt \\
&= 2\left(\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right) + \frac{e^{-st}}{-s}\bigg|_0^1 + 3\left(\frac{e^{-st}}{-s}\right)\bigg|_1^\infty \\
&= 2\left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) + \frac{e^{-s}}{-s} - 2\left(0 - \frac{1}{s^2}\right) + \frac{1}{s} + 3\left(0 + \frac{e^{-3s}}{s}\right) \\
&= \frac{-e^{-s}(3s+2) + 2 + s + 3se^{-3s}}{s^2}
\end{aligned}$$

Practice Problems.

a. Use the definition of Laplace transforms to find some of the common Laplace transform formulas.
   - i. $\mathcal{L}\{e^{-3t}\}$
   - ii. $\mathcal{L}\{e^{-at}\}$
   - iii. $\mathcal{L}\{\sin(t)\}$
   - iv. $\mathcal{L}\{\sin(2t)\}$
   - v. $\mathcal{L}\{\cos(t)\}$
   - vi. $\mathcal{L}\{\sinh(t)\}$ [Hint: it might be easier to convert this to the exponential definition to do the integration. Then simplify after integrating.]
   - vii. $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 0, 0 \le t \le 1 \\ 1, \quad t \ge 1 \end{cases}$
   - viii. $\mathcal{L}\{t^3\}$ [Hint: tabular method for by-parts works great here.]

b. Use the definition of Laplace transforms to find some more complicated Laplace transform formulas. We'll see how to build these from basic components later, for now, find them by integrating.
   - i. $\mathcal{L}\{e^{t+5}\}$
   - ii. $\mathcal{L}\{2t^2 - 3t + 4\}$
   - iii. $\mathcal{L}\{tsin(t)\}$ [Hint: Let u=t, and $dv = \sin(t)e^{-st}$, and then integrate dv by parts.]
   - iv. $\mathcal{L}\{e^t \sin(t)\}$
   - v. $\mathcal{L}\{e^{-t}\cosh(t)\}$ [Hint: it may help to use the definition of cosh(t) here to make the integration easier.]
   - vi. $\mathcal{L}\{cos^2(t)\}$

2. Common Formulas

In practice, we don't do the integration for Laplace transforms unless we really need to. Instead, we work from a table of common transform formulas. The basic formulas we'll be working from are listed below, but you can't find tables with many more formulas.

A. $\mathcal{L}\{1\} = \frac{1}{s}$

B. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

E. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n=1,2,3...

F. $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

C. $\boldsymbol{\mathcal{L}}\{\cos kt\} = \frac{s}{s^2+k^2}$

D. $\boldsymbol{\mathcal{L}}\{\cosh kt\} = \frac{s}{s^2-k^2}$

G. $\boldsymbol{\mathcal{L}}\{\sinh kt\} = \frac{k}{s^2-k^2}$

H. $\boldsymbol{\mathcal{L}}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$

Incidentally, the unit step function $\mathcal{U}(t-a) = \begin{cases} 0, 0 \le t \le a \\ 1, \quad t \ge a \end{cases}$.

Let's use these common formulas to convert functions into their Laplace transforms.

Example 4. Find $\boldsymbol{\mathcal{L}}\{e^{-2t}\}$.

This function corresponds to formula B, with a= (-2). To find F(s) just plug in where a goes in the transform.

$$\boldsymbol{\mathcal{L}}\{e^{-2t}\} = \frac{1}{s-(-2)} = \frac{1}{s+2}$$

Example 5. Find $\mathcal{L}\{2t^2 - 3t + 4\}$.

We did this one by integration already in the practice problem set, but let's try to come up with the answer just using the basis transform formulas.

$$\mathcal{L}\{2t^2 - 3t + 4\} = 2\mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} + 4\mathcal{L}\{1\} = 2\left(\frac{2!}{s^3}\right) - 3\left(\frac{1!}{s^2}\right) + 4\left(\frac{1}{s}\right) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{4}{s}$$

Example 6. Find $\mathcal{L}\{\sin(5t) + \cos(2t)\}$

Breaking this up term-by-term, we get $\mathcal{L}\{\sin(5t) + \cos(2t)\} = \mathcal{L}\{\sin(5t)\} + \mathcal{L}\{\cos(2t)\}$. For the sine function, k=5. For the cosine function, k=2. Plugging these into the Laplace transform formulas, we get $\frac{5}{s^2+25} + \frac{s}{s^2+4}$.

Practice Problems.

c. Use our list of common formulas to find the Laplace transform of the following functions. Be sure to list which formula you used.

- i. $\mathcal{L}\{e^{t+5}\}$
- ii. $\mathcal{L}\{(3t-1)^2\}$
- iii. $\mathcal{L}\{sin(t)\cos(t)\}$ [Hint: Use an identity first.]
- iv. $\mathcal{L}\{e^{-t}\cosh(t)\}$ [Hint: it may help to use the definition of cosh(t) hereto complete the transform.]
- v. $\mathcal{L}\{cos^2(t)\}$
- vi. $\mathcal{L}\{12t^5\}$
- vii. $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 0, & 0 \le t \le 3 \\ 7, & t \ge 3 \end{cases}$

3. Inverse Laplace Transforms

We'll use the same list of common formulas to do inverse Laplace transforms. There is an integration formula for it, but it depends on complex numbers, which is beyond what we are trying to accomplish here. Instead, we are going to start with F(s) and try to match it to the right side of the formulas in our list—maybe doing some algebra to do it—and then convert back to the f(t) function that goes with it. The notation for the inverse Laplace transform is $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Example 7. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$.

Going back to our list of formulas, the equation this looks most like is formula E: $\boldsymbol{\mathcal{L}}\{t^n\} = \frac{n!}{s^{n+1}}$. To use this formula, we need to find out what n is first. The power in the denominator is n+1=3, which means n=2. To match the formula exactly, we need the numerator to be 2!=2. We can multiple/divide our equation by a constant to get the match we need:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$$

By multiplying on the inside by 2, and the outside by ½ we aren't changing the equation, but we can make the inside of the expression look like what we need. Applying the t^n formula, we get $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$.

Example 8. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$.

Often, we'll have to do algebra before we can perform the inverse operation. One of the most important skill will be partial fractions, which is what we'll need here.

$$\frac{1}{s^2+3s} = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{A(s+3)+Bs}{s(s+3)}$$

This means that $1 = A(s+3) + Bs$.

We can choose values for s to solve for A and B.

When s=0, the equation reduces to 1=3A, so A=1/3. When s=-3, the equation reduces to 1=B(-3), or B=(-1/3).

That means that:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+3}\right\} = A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$
$$= \frac{1}{3}(1) - \frac{1}{3}e^{-3t} = \frac{1}{3} - \frac{e^{-3t}}{3}$$

For our formula for the second term (Formula B), note that the a=(-3) because you always have to flip the sign of the constant because of the negative in the formula.

If you are stuck with finding the constants, you can do the transform before finding them. When the problems get longer, finding all the constants can take time and be tedious. This way, you'll have the general form of the equation.

Practice Problems.

d. Find the inverse transforms of the following functions in the s-domain using the list of standard formulas from our table. You'll need to do algebra and split them up into terms (in most cases) in order to match equations in the list.

   i. $\mathcal{L}^{-1}\left\{\left(\frac{2}{s}-\frac{1}{s^3}\right)^2\right\}$
   ii. $\mathcal{L}^{-1}\left\{\frac{1}{5s-2}\right\}$ [Hint: factor out the 5 from the denominator.]
   iii. $\mathcal{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$
   iv. $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$
   v. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$
   vi. $\mathcal{L}^{-1}\left\{\frac{s-3}{s^2-3}\right\}$
   vii. $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\}$
   viii. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$
   ix. $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{5s}\right\}$

4. Translations

We have two "translation" formulas. One is a translation in the s-domain, and one is a translation in the t-domain (and is related to the unit step function we mentioned earlier).

The first translation formula is $\mathcal{L}\{e^{bt}f(t)\} = F(s-b)$.

The second translation formula is $\mathcal{L}\{f(t-b)\mathcal{U}(t-b)\} = e^{-bs}F(s)$.

It's easy to confuse these formulas because both have e^{bt} or something like it in the formulas, but it matters which side of the equation it's on.

The first translation formula acts a bit like a substitution in calculus, especially when doing the inverse Laplace transform.

Example 9. Find $\mathcal{L}\{t^2e^{-t}\}$.

The translation here is the e^{-t}, with b=(-1). The rest of the expression we want to transform is t^2. The Laplace transform of that is $\frac{2!}{s^3}$. For the translation, we'll replace s with s-(-1)=s+1. So finally, we get:

$$\mathcal{L}\{t^2e^{-t}\} = \frac{2!}{(s+1)^3}$$

Example 10. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$.

To complete the inverse here, we can't factor the denominator, so instead, we'll complete the square.

$$\frac{1}{s^2+4s+5}=\frac{1}{(s^2+4s+4)+1}=\frac{1}{(s+2)^2+1}$$

The translation is s+2, which makes b=(-2); this goes into the exponential component. Looking at the form of the problem without the translation we have $\frac{1}{s^2+1}$, which is a sine function with k=1.

Thus we get:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}\sin(t)$$

The second translation formula is used for transforming piecewise functions.

Example 11. Find the $\mathcal{L}\{f(t)\}$ for $f(t)=\begin{cases}1, 0\le t\le 2\\ -1, \quad t\ge 2\end{cases}$.

For any general piecewise function of the form: $f(t)=\begin{cases}g(t), 0\le t\le b\\ h(t), \quad t\ge b\end{cases}$, we can rewrite the equation in terms of the unit step function: $f(t)=g(t)-g(t)\mathcal{U}(t-b)+h(t-b)\mathcal{U}(t-b)$. In this case, g(t)=1, and h(t)=(-1). So, in terms of the unit step function we have: $f(t)=1-(1)\mathcal{U}(t-2)+(-1)\mathcal{U}(t-2)=1-\mathcal{U}(t-2)-\mathcal{U}(t-2)=1-2\mathcal{U}(t-2)$.

Thus:

$$\mathcal{L}\{f(t)\}=\mathcal{L}\{1-2\mathcal{U}(t-2)\}=\mathcal{L}\{1\}-2\mathcal{L}\{\mathcal{U}(t-2)\}=\frac{1}{s}-2\frac{e^{-2t}}{s}$$

The most potentially problematic term in the process is $g(t)\mathcal{U}(t-b)$, since our translation formula works when both parts of the product are in terms of (t-b). You may need to apply identities to the g(t) function to get it in terms of (t-b). This isn't a concern when g(t) is a constant (since there is no t-dependence), and functions of t can be rewritten. For instance, t^2+4 can be rewritten as follows:

Supple you replace t^2 with $(t-b)^2$. What have you done to the expression? $(t-b)^2=t^2-2bt+b^2$. So you've added -2bt and b^2. Subtract them back off.

$$t^2+4=(t-b)^2+2bt-b^2+4$$

Repeat this process for the t term. Replace 2bt with 2b(t-b). What have you changed the expression by? $2b(t-b)=2bt-2b^2$, so we have an extra $-2b^2$. Subtract that off.

$$t^2+4=(t-b)^2+2b(t-b)+2b^2-b^2+4=(t-b)^2+2b(t-b)+(b^2+4)$$

Now that your expression is in terms of (t-b), you can perform the transform.

Some functions will be easier. For instance, $\sin(t) = \sin(t \pm 2\pi)$, so if $b = \pm 2\pi$, this is easy to fix. You may also need to apply properties of even and odd functions together with the co-function identities if you are working with $\frac{\pi}{2}$.

Or you can use an alternative version of the second translation formula which states that $\mathcal{L}\{g(t)\mathcal{U}(t - b) = e^{-bs}\mathcal{L}\{g(t + a)\}$.

Example 12. Find $\mathcal{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+3}\right\}$.

For this problem we first need to multiply out the numerator and separate the expression into several terms.

$$\frac{(1 + e^{-2s})^2}{s + 3} = \frac{1 + 2e^{-2s} + e^{-4s}}{s + 3} = \frac{1}{s + 3} + 2\left(\frac{e^{-2s}}{s + 3}\right) + \frac{e^{-4s}}{s + 3}$$

Taking each term one at a time:

$\frac{1}{s+3}$: This is straight forward. There are no translation components here, so since a=(-3) we get $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$ from Formula B.

$\frac{e^{-2s}}{s+3}$: The e^{-2s} component here is the unit step translation, with b=2, and the denominator, as with the first term tells us that a=(-3) from formula B. Putting these together we get $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} = e^{-3t}\mathcal{U}(t - 2)$.

$\frac{e^{-4s}}{s+3}$: The e^{-4s} component is the unit step translation, with b=4. The numerator is the same basic exponential function in the previous two terms with a=(-3). Putting these together we get $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s+3}\right\} = e^{-3t}\mathcal{U}(t - 4)$.

Thus:

$$\mathcal{L}^{-1}\left\{\frac{(1 + e^{-2s})^2}{s + 3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s + 3} + 2\left(\frac{e^{-2s}}{s + 3}\right) + \frac{e^{-4s}}{s + 3}\right\} = e^{-3t} + 2e^{-3t}\mathcal{U}(t - 2) + e^{-3t}\mathcal{U}(t - 4)$$

Practice Problems.

e. Perform the indicated transform or inverse transform using the basic formulas together with the translation formulas.
   i. $\mathcal{L}\{e^{2t}(t - 3)^2\}$
   ii. $\mathcal{L}\{e^{-4t}\sin(5t)\}$
   iii. $\mathcal{L}\{(t - 1)\mathcal{U}(t - 1)\}$
   iv. $\mathcal{L}\{(3t + 1)\mathcal{U}(t - 2)\}$
   v. $\mathcal{L}\left\{\sin(t)\,\mathcal{U}\left(t - \frac{\pi}{2}\right)\right\}$
   vi. $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} t, 0 \le t \le 1 \\ 1, \quad\;\; t \ge 1 \end{cases}$

vii. $\mathcal{L}\{g(t)\}$ for $g(t) = \begin{cases} \sin(t), 0 \le t \le \pi \\ cost, \quad\quad t \ge \pi \end{cases}$

viii. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$

ix. $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$

x. $\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$

xi. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

xii. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$

xiii. $\mathcal{L}^{-1}\left\{\frac{se^{-s}}{s^2+49}\right\}$

5. Derivatives and Integrals

There are a number of derivative and integral formulas for Laplace transforms. Like our translation formulations, there are versions with derivative of f(t) and F(s). Our integration formulas come in two main types: one for convolutions (which we'll talk about), and transforms of integrals.

The first derivative formula is $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$, where $F(s) = \mathcal{L}\{f(t)\}$.

We can apply this by comparing the results in Example 9 for $\mathcal{L}\{t^2 e^{-t}\}$. There we used the translation formula on $\mathcal{L}\{t^2\}$, here we're going to use the derivative formula on $\mathcal{L}\{e^{-t}\}$.

Example 13. Find $\mathcal{L}\{t^2 e^{-t}\}$.

Our $f(t) = e^{-t}$. So $\mathcal{L}\{e^{-t}\}$, with b=(-1), becomes $F(s) = \frac{1}{s+1}$. Now, we need to take the derivative of this (with respect to s) twice.

$$\frac{d}{ds}\left[\frac{d}{ds}((s+1)^{-1})\right] = \frac{d}{ds}[-(1)(s+1)^{-2}] = (-1)(-2)(s+1)^{-3} = \frac{(-1)^2 2!}{(s+1)^3}$$

Finally, multiply by $(-1)^2$: $\frac{(-1)^2(-1)^2 2!}{(s+1)^3} = \frac{2!}{(s+1)^3}$

This is the same answer we had in Example 9. You'll notice that the $(-1)^n$ is just there to make the result the same sign as the original F(s).

The second derivative formula is for transforms of derivatives.

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

To make this a little more concrete, let's look at the first couple derivative formulas.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

We will use these formulas to solve differential equations. So let's just try to apply the Laplace Transform to the equation. We'll solve it the rest of the way later on.

Example 14. Find the Laplace transform of the differential equation $y'' + 2y' + y = t, y(0) = 1, y'(0) = 0$.

We'll use Y(s) to indicate the Laplace transform of y(t).

$$\mathcal{L}\{y''^{(t)}\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 0$$
$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1$$
$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'' + 2y' + y\} = s^2Y(s) - s + 2(sY(s) - 1) + Y(s)$$
$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$s^2Y(s) - s + 2sY(s) - 2 + Y(s) = \frac{1}{s^2}$$

So, we need to collect the Y(s) terms together and solve for it. We'll do that later on.

We have two integral formulas. One is for directly transforming an integral. $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$, where tau is a dummy variable.

Example 15. Find $\mathcal{L}\left\{\int_0^t \tau\sin(\tau)d\tau\right\}$.

To find the transform we'll start with $f(t) = tsin(t)$. This will use one of our derivative formulas on sin(t).

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1}$$

To find $\mathcal{L}\{\text{tsin}(t)\}$ we'll use our derivative formula: $\mathcal{L}\{\text{tsin}(t)\} = (-1)\frac{d}{ds}(s^2 + 1)^{-1} = (-1)(-1)(s^2 + 1)^{-2}(2s) = \frac{2s}{(s^2+1)^2}$.

Our integration formula says that the Laplace transform of this integral is this function that we just found divided by s.

$$\frac{F(s)}{s} = \frac{\frac{2s}{(s^2 + 1)^2}}{s} = \frac{2}{(s^2 + 1)^2}$$

You can check this by doing the integration, and finding the transform of that function.

The second type of integration formula is for a function called a convolution. A convolution is defined as $f * g = \int_0^t f(\tau)g(t-\tau)d\tau$ where tau again is dummy variable for the integration. The nice thing about the convolution is that it produces a nice Laplace transform:

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

Example 16. Find $\mathcal{L}\{t * \sin(t)\}$.

Not that this problem is different than in Example 15, and different than $\mathcal{L}\{t\sin(t)\}$. This * symbol is for the convolution, not the product.

Applying the transform of a convolution formula, we need $\mathcal{L}\{t\}\mathcal{L}\{\sin(t)\} = \frac{1}{s^2}\left(\frac{1}{s^2+1}\right) = \frac{1}{s^2(s^2+1)}$.

Practice Problems.

f. Perform the transforms or inverse transforms as indicated.
   - i. $\mathcal{L}\{te^{-3t}cos4t\}$
   - ii. $\mathcal{L}\{tsinh(t)\}$
   - iii. $\mathcal{L}\{t^2\sin(t)\}$
   - iv. $\mathcal{L}\{t^2 * \cos(2t)\}$
   - v. $\mathcal{L}\{e^{5t} * \cosh(t)\}$
   - vi. $\mathcal{L}\{t * e^{4t}\}$
   - vii. $\mathcal{L}\left\{\int_0^t \sin\tau\, d\tau\right\}$
   - viii. $\mathcal{L}\left\{t\int_0^t \cosh\tau\, d\tau\right\}$
   - ix. $\mathcal{L}\left\{\int_0^t \tau^3\sin(\mathrm{t}-\tau)\, d\tau\right\}$
   - x. $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}$
   - xi. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$
   - xii. $\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2}\right\}$ [Hint: try using the derivative formulas by integrating with respect to s.]
   - xiii. $\mathcal{L}^{-1}\left\{\frac{s}{s^2(s^2+4)}\right\}$

g. Perform the transform or inverse transforms as indicated. [These problems contain problems from all the previous sections.
   - xiv. $f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$
   - xv. $f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$
   - xvi. $f(t) = te^{4t}$
   - xvii. $f(t) = t^5$
   - xviii. $f(t) = \cos 5t + \sin 2t$
   - xix. $f(t) = \cosh 4t$
   - xx. $f(t) = e^t sin3t$
   - xxi. $f(t) = e^{2t}(t-1)^2$
   - xxii. $f(t) = t^2 * te^t$
   - xxiii. $f(t) = \int_0^t \tau\cos\tau\, d\tau$

xxiv. $f(t) = \int_0^t \tau e^{t-\tau} d\tau$
xxv. $F(s) = \frac{1}{s^2}$
xxvi. $F(s) = \frac{1}{4s+1}$
xxvii. $F(s) = \frac{5}{s^2+49}$
xxviii. $F(s) = \frac{2s-6}{s^2+9}$
xxix. $F(s) = \frac{s}{s^2+2s-3}$
xxx. $F(s) = \frac{2s-4}{(s^2+s)(s^2+1)}$
xxxi. $F(s) = \frac{1}{s^2+2s+5}$ (hint: complete the square)
xxxii. $F(s) = \frac{(s+1)^2}{(s+2)^4}$
xxxiii. $F(s) = \frac{1}{s(s-1)}$

6. Solving Differential Equations.

We performed a transform on a differential equation in Example 14. We'll finish it here.

Example 17. Find the solution to the differential equation $y'' + 2y' + y = t, y(0) = 1, y'(0) = 0.$

Earlier, we found that the Laplace transform of this equation is

$$s^2Y(s) - s + 2sY(s) - 2 + Y(s) = \frac{1}{s^2}$$

Collect the terms on left containing Y(s) and all the others put on the right.

$$s^2Y(s) + 2sY(s) + Y(s) = \frac{1}{s^2} + (s+2)$$
$$(s^2 + 2s + 1)Y(s) = \frac{1}{s^2} + (s+2)$$
$$Y(s) = \frac{1}{s^2(s+1)^2} + \frac{s+2}{(s+1)^2}$$

From here, we have some options. We can use any one of our formulas. Certainly, we could use partial fractions. Or we could use the convolution formulas for the first term, and maybe a derivative formula on the second term, together with some algebra. Any way of getting this back into the t-domain is okay.

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)\left(\frac{1}{(s+1)^2}\right)\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)\right\}\mathcal{L}^{-1}\left\{\left(\frac{1}{(s+1)^2}\right)\right\}$$

The second function in the product is Formula E. The second one is the same formula together with a translation formula.

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)\right\}\mathcal{L}^{-1}\left\{\left(\frac{1}{(s+1)^2}\right)\right\} = t * te^{-t}$$

For the second term:

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1+1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$
$$= e^{-t} + te^{-t}$$

So, our answer is $y(t) = t * te^{-t} + e^{-t} + te^{-t}$ in terms of the convolution, which we could integrate. Alternatively, partial fractions would have allowed us to skip the convolution, but I'll leave it here.

Practice Problems.

h. Solve the differential equations by taking the Laplace transform. Be sure to perform an inverse transform to get the solution back in the t-domain.
   i. $y' - y = 1, y(0) = 0$
   ii. $y' + 6y = e^{4t}, y(0) = 2$
   iii. $y'' + 5y' + 4y = 0, y(0) = 1, y'(0) = 0$
   iv. $y'' - 4y' = 6e^{3t} - 3e^{-t}, y(0) = 1, y'(0) = -1$
   v. $y'' - 4y' + 4y = t^3e^{2t}, y(0) = 0, y'(0) = 0$
   vi. $y'' - y' = e^t \cos t\,, y(0) = 0, y'(0) = 0$
   vii. $y' + 2y = f(t), y(0) = 0, f(t) = \begin{cases} t, 0 \le t < 1 \\ 0, \quad t \ge 1 \end{cases}$
   viii. $y'' + 4y = \sin t\, \mathcal{U}(t - 2\pi), y(0) = 1, y'(0) = 0$
   ix. $y'(t) = 1 - \sin t - \int_0^t y(\tau)d\tau\,, y(0) = 0$