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How to recognize the different types of differential equations

Figuring out how to solve a differential equation begins with knowing what type of differential equation it is. Since there is no "one way" to solve them, you need to know the type to know the solution method needed for that equation. This handout will cover a variety of differential equation types and attempt to categorize them so that you can better understand what to do when getting started solving an equation and are not told the method to use up front. We will not cover any of the solution techniques here, however.

1. Ordinary or Partial?

The main thing to look for in determining whether a differential equation is ordinary or partial is the derivative notation used.

Ordinary differential notation: $\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ dy,\ dx,\ dt,\ \frac{dy}{dt},\ y',\ y'', y''', \dot{y},\ \ddot{y},\ \dddot{y}$

Partial differential notation: $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, u_x, u_t, u_{xx}, u_{xy}, u_{zz}$

Practice Problems.

I. Determine if the following equations are ordinary or partial.

a. $\frac{dy}{dt} + ty^2 = 0$
b. $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sint$
c. $y''' + ty' + (cos^2 t)y = t^3$
d. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
e. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$
f. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
g. $\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

2. Order

The next thing to determine is the order of the equation. The order depends on the highest derivative term in the equation (for any variable or combination of variables).

First Order: $\frac{dy}{dx},\ dy,\ dx,\ dt,\ \frac{dy}{dt},\ y', \dot{y},\ \frac{\partial u}{\partial x}, u_x, u_t$

Second Order: $\frac{d^2y}{dx^2},\ y'', \ddot{y},\ \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, u_{xx}, u_{xy}, u_{zz}$

Third and higher Orders: $\frac{d^3y}{dx^3}, y''',\ y^{IV},\ y^{(4)},\ \dddot{y},\ \frac{\partial^4 u}{\partial x^2 \partial y^2},\ u_{ttt},\ u_{xyz}, u_{zzz}$

Practice Problems.

II. Determine the order of the following equations.

h. $\frac{dy}{dt} + ty^2 = 0$

i. $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sint$

j. $y''' + ty' + (cos^2 t)y = t^3$

k. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

l. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$

m. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

n. $\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

3. Linearity

Linearity is a property of differential equations that relates to the relationship of the function to its derivatives. For our purposes, linearity is not affected by anything happening to the independent variable; in ordinary differential equations this is typically x or t.

Linear terms: $t,\ y,\ y',\ \mathrm{t}y', t^2y'', \sin(t)\, y, e^t\dot{y}, \ln(x)\, y''', u_{tt}$

Nonlinear terms: $y^2, (y'')^3, yy', \sin(y)\,, e^y, uu_x,\ u_t u_y$

Practice Problems.

III. Determine whether the following equations are linear or nonlinear.

o. $\frac{dy}{dt} + ty^2 = 0$

p. $t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = sint$

q. $y''' + ty' + (cos^2 t)y = t^3$

r. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

s. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$

t. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

u. $\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

IV. Put it all together now. Determine linearity, order and ordinary or partial for each equation. Compare your answers to your previous results.

Once we've gotten the basic type, then we can break things down a little further. I'll deal with ordinary only here, since there are many more solvable types that need to be differentiated.

4. Higher Order types

Higher order problems are tricky to solve, so we will look at two main types: those with constant coefficients and those with polynomial coefficients (polynomials in the independent variable). We will assume that we are just considering the homogeneous type higher-order equations (all the terms include the function variables and its derivatives), since we will need the solutions to the homogeneous form to solve for the remaining forcing term (such techniques are dealt with elsewhere).

A. Constant Coefficients

In this type of problem, the equation is linear, and the functions of the independent variable that multiply the function we are trying to solve for and its derivatives are strictly constants and do not depend on the independent variable.

For instance, $ay'' + by' + cy = 0$, or $ay''' + by'' + cy' + dy = 0$

For such equations we assume a solution of the form $y = e^{rt}$ or $y = e^{rx}$. This will give a characteristic equation you can use to solve for the values of r that will satisfy the differential equation.

B. Polynomial Coefficients

If the coefficients are polynomials, we could be looking at either a Cauchy-Euler equation, or a series solution problem. To be Cauchy-Euler, the polynomial coefficients need to be of a specific type: they need to increase by a single power for each derivative taken.

Cauchy-Euler: $t^2y'' + 3ty' - 5y = 0$ or $x^5y''' - 2x^4y'' + 6x^3y' + x^2y = 0$

Series Solutions: $(x^2 + 1)y'' - 2xy' - 3y = 0$ or $xy'' - 2x^3y' + 8y = 0$

If you have a Cauchy-Euler equation, assume the solution $y = t^n$ or $y = x^n$. This will produce an auxiliary equation you can use to solve for n.

If you have a series solution equation, assume $y = \sum_{n=1}^{\infty} c_n(x - x_0)^n$ or $y = \sum_{n=1}^{\infty} c_n(x - x_0)^{n+r}$. The former if you are expanding around an ordinary point, and the second version if you are expanding around a regular singular point (Frobenius Theorem).

If you have a non-polynomial coefficient, you may be able to approximate the solution by using the first couple terms of the Taylor series.

Practice Problems.

V. Determine the assumed solution for the higher order differential equations below.

v. $y'' + 2y' = 0$
w. $y'' - y' - 2y = 0$
x. $y'' + y = 0$
y. $y'' + 4y = 0$
z. $(1-t)y'' + ty' - y = 0$
aa. $x^2y'' - 3xy' + 4y = 0$
bb. $y'' - y = 0, y(t) = 0$
cc. $y^{IV} + y''' + 3y = t, y(t) = 0$
dd. $x^2y'' - 2y = 0$
ee. $x^2y'' + 5xy' + 4y = 0$
ff. $x^3y''' - 6y = 0$
gg. $4y'' + 12y' + 9y = 0$

hh. $25y'' - 20y' + 4y = 0$
ii. $y'' + xy' + 2y = 0, x_0 = 0$
jj. $xy'' + y' + xy = 0, x_0 = 1$
kk. $(x+3)^2 y'' - y = 0$
ll. $y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$
mm. $2xy'' - y' + 2y = 0$
nn. $2x^2y'' - xy' + (x^2+1)y = 0$
oo. $xy'' = xy' + y = 0$

5. First Order Equations

There are many more options for solving first order equations since there is only one derivative involved, and because of that, there are many more specific types of equations, and many more possibilities that need to be checked. It should be noted that the simplest equations of this form can be solved by the same methods as higher order equations if the conditions hold, but this is generally not the way first order equations are taught because they are too restrictive and solve very few equations.

First order equations tend to come in two primary forms: $M(x,y)dx + N(x,y)dy = 0$ or $y' = f(x,y)$. All equations can be written in either form, but equations can be split into two categories roughly equivalent to these forms. For instance: Separable, Homogeneous and Exact equations tend to be in the differential form (former), while Linear, and Bernoulli tend to be in the latter. However, since simple algebra can get you from one form to another, the crucial feature is really the type of function f(x,y) you obtain. If it can be reduced to obtain a single linear y term (and possibly a polynomial y term), then it might be linear or Bernoulli. Both of these types are aided by getting the equation into standard form: $y' + p(x)y = g(x,y)$.

Parts A-C below will consider the first branch (the non-simplifying branch). Parts D-E will consider the second branch. Part F will remind you want to do if none of these apply, because there are plenty of differential equations that cannot be solved analytically.

A. Separable Equations

Separable equations can be determined by only be determined by performing algebra on a problem. One must be able to get all the y terms on one side, dy in the numerator and dy must multiply all the terms on that side so that it can be integrated. Similarly, the other side of the equation must contain all the independent variables (x or t) and dx (or dt) in the numerator, and all the x (or t) terms multiplied by the dx (or dt).

Example 1. Show that the equation $(y^2+1)dx = ysec^2xdy$ is separable.

If we divide both sides by y^2+1 and sec^2x we get $\frac{dx}{sec^2x} = \frac{ydy}{y^2+1}$. These can be easily integrated if we convert the secants to cosines.

Example 2. Show that the equation $(e^x \sin y + 3y)dx + (3x + e^x \cos y)dy = 0$ is not separable.

In this case we can try dividing, but both terms include both x and y, and there is no way to factor out on either side in a way that we can isolate the variables. That means this equation is not separable.

B. Homogeneous Equations

Sometimes we can made non separable equations into separable ones with the substitution y=vx, where v is some function of x. To test whether this will work or not, replace x with tx, and y with ty. If you can factor out all the t's from each term of the equation, then it will be homogeneous and this substitution will work.

Example 3. Determine if the equation $(x - y)dx + xdy = 0$ is homogeneous.

If we do the replacement we get $(tx - ty) = t(x - y)$, and obviously, this works fine with tx = t(x). From here, do the replacement y=vx (and its derivative). The equation will be separable now.

C. Exact

Like the two previous examples, exact equations tend to come in the differential form: $M(x, y)dx + N(x, y)dy = 0$. However, these equations are neither separable nor homogeneous. (Yes, test these first.) To determine if an equation is exact check the following relation: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. If this holds, then the equation is exact, and proceed to find a find ψ(x,y) by integrating M with respect to x, and N with respect to y, much as you would find a potential function.

If the equations are not exact in their present form, they might still be exact with the use of an integrating factor. Check the following equations: $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$ or $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$. These must be functions of a single variable. But multiplying the original equation by these factors will result in an equation that is exact and can be solved as above.

Example 4. Determine if the equation $(2x - y)dx + (2y - x)dy = 0$ is exact.

Here M=2x-y and N=2y-x. If we differentiate M with respect to y we get -1. If we differentiate N with respect to x we get -1. This equation is exact.

Example 5. Determine if the equation $ydx + (2xy - e^{-2y})dy = 0$ is exact.

Here M = y, and N = $2xy - e^{-2y}$. The derivative of M with respect to y is 1. And the derivative of N with respect to x is 2y. That's not the same, so check the integrating factors.

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{1-2y}{2xy - e^{-2y}} dx}$$

This is not a function of one variable.

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{2y-1}{y} dy} = e^{\int 2 - \frac{1}{y} dy} = e^{2y - lny} = e^{2y} e^{\ln(\frac{1}{y})} = \frac{e^{2y}}{y}$$

That is a function of one variable, so if we multiply, our original equation becomes: $e^{2y}dx + \left(2xe^{2y} - \frac{1}{y}\right)dy = 0$.

Now if we check our new M and N, we get $M_y = 2e^{2y}$, and $N_x = 2e^{2y}$. Those agree and so ψ(x,y) does exist.

Practice Problems.

VI. Determine if the equations are exact. If they are not exact as they are, check for an integrating factor.

pp. $\left(\frac{y}{x} + 6x\right)dx + (\ln x - 2)dy = 0$
qq. $(x + y)^2 dx + (2xy + x^2 - 1)dy = 0$
rr. $(4y + 2t - 5)dt + (6y + 4t - 1)dy - 0$
ss. $(y^2 \cos x + 3x^2 y - 2x)dx) + (2y \sin x - x^3 + lny)dy = 0$
tt. $y' = e^{2x} + y - 1$
uu. $6xydx + (4y + 9x^2)dy = 0$
vv. $(10 - 6y + e^{-3x})dx + 2dy = 0$
ww. $(x^2 + y^2 - 5)dx = (y + xy)dy$

D. Linear Equations

Linear equations can be put into standard form: $y' + p(x)y = g(x)$. If the equation is in differential form, you'll have to do some algebra. If you can't get it to look like this, then the equation is not linear. If it is linear, it can be solved either by an integrating factor used to turn the left side of the equation into a product rule, or the formula derived from this procedure.

Practice Problems.

VII. Show that the following equations are linear. Find p(x) and g(x) (or p(t) and g(t)).

xx. $y' + \frac{2}{t}y = \frac{\cos t}{t^2}, y(\pi) = 0, t > 0$
yy. $ty' + (t + 1)y = t, y(ln2) = 1, t > 0$
zz. $3\frac{dy}{dx} + 12y = 4$
a. $x\frac{dy}{dx} - y = x^2 \sin(x)$
b. $y' = t + 2y$
c. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$

E. Bernoulli Equations

Bernoulli equations look superficially like linear equations in that the right side can be made to look like $y' + p(x)y$, but the right side of the equation isn't only a function of x, but rather $g(x)y^n$, where n is some number which is neither 0 nor 1 (when it is 0 or 1 it reduces to either a separable or linear case). If the problem is Bernoulli, there is a substitution procedure involving $z = y^{1-n}$ that will also you to convert the problem to a linear equation in z and solve it.

Practice problems.

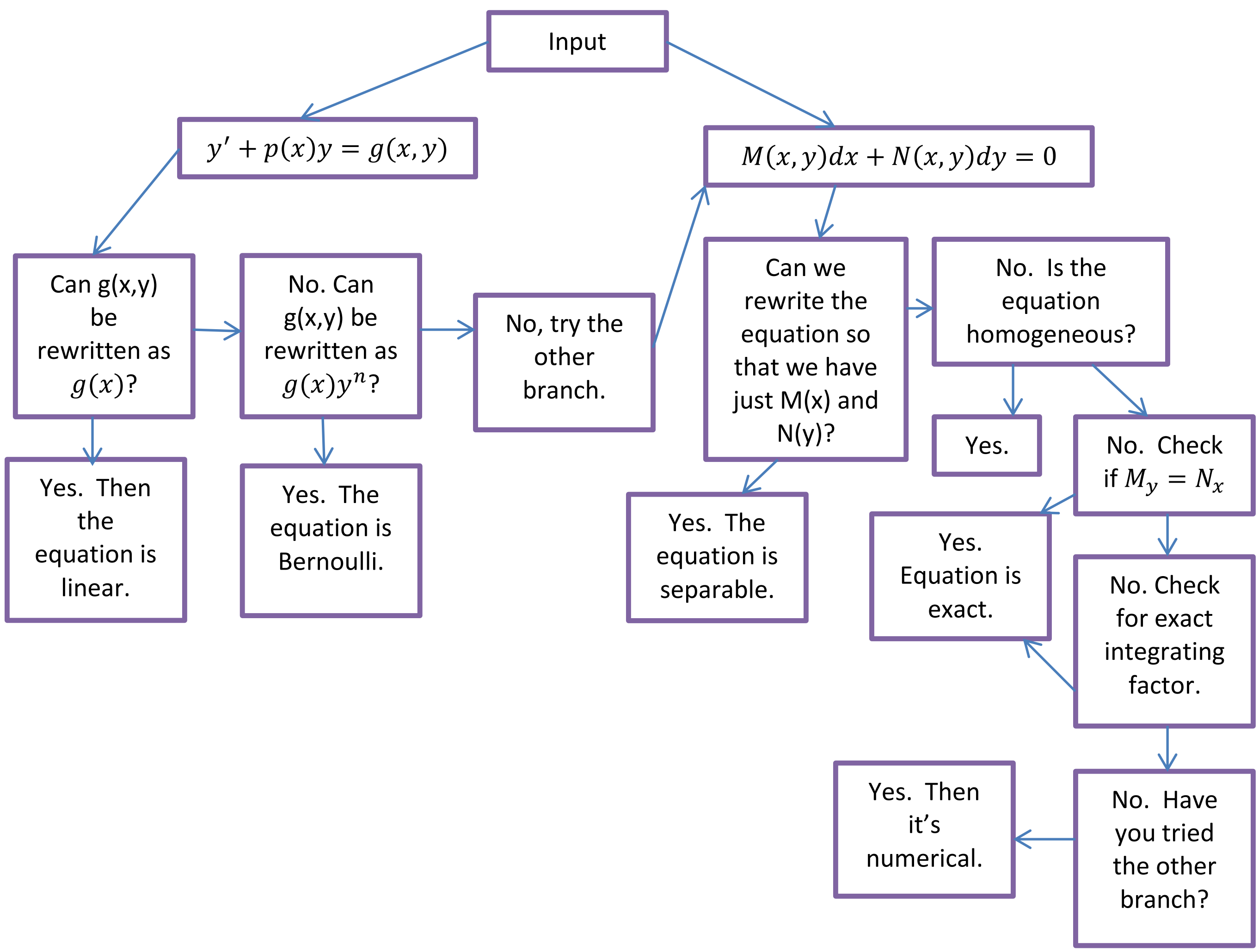
VIII. Show that the following equations are Bernoulli. State the substitution you'd use.

d. $y' + 2xy = xy^2$
e. $xy' + y = xy^3$
f. $yy' + \frac{1}{x}y^2 = x\sqrt{y}$
g. $x\frac{dy}{dx} + y = \frac{1}{y^2}$
h. $\frac{dy}{dx} = y(xy^3 - 1)$

F. Other

If you can't rearrange your equation to look like one of these types, it may not be solvable analytically by known techniques. You can use numerical methods to approximate a solution.

When deciding which of these cases applies, first divide it into two cases, then in each branch, do a series of checks until you hit on something.



Some equations can be done in more than one way, but one way may be easier than another. It may be beneficial to check for another method to see which takes less work.

Practice Problems.

IX. Determine which solution method would be used for each of the following first order equations.

i. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$
ii. $y' + 3x^2y = x^2y^3$
iii. $y' = \frac{x^2+y^2}{2xy}$
iv. $(e^x \sin y + 3y)dx + (3x + e^x \cos y)dy = 0$
v. $\sin 2x\, dx + \cos 3y\, dy = 0$
vi. $y' + y = te^{-t} + 1$
vii. $yy' - 2y^2 = e^x$
viii. $xy' = (1-y^2)^{1/2}$
ix. $-y^2dx + x(x+y)dy = 0, y(1) = 1$
x. $(y^2+1)dx = ysec^2xdy$
xi. $y(lnx - lny)dx = (xlnx - xlny - y)dy$
xii. $(6x+1)y^2\frac{dy}{dx} + 3x^2 = 2y^3$
xiii. $\frac{dx}{dy} = -\frac{4y^2+6xy}{3y^2+2x}$
xiv. $(2x+y+1)y' = 1$
xv. $(x^2+4)dy = (2x - 8xy)dx$
xvi. $\frac{y}{x^2}\frac{dy}{dx} + e^{2x^3+y^2} = 0$
xvii. $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1$