

Differential Equations Characteristic Equations Handout Key

1. $y' + 2y = 0$ $y = e^{rt}$ $y' = re^{rt}$ $re^{rt} + 2e^{rt} = e^{rt}(r+2) = 0$
 $\Rightarrow r = -2$ $y(t) = c_1 e^{-2t}$

2. $2y' - 3y = 0$ $2r - 3 = 0 \Rightarrow r = \frac{3}{2}$ $y(t) = c_1 e^{\frac{3}{2}t}$

3. $y'' - 7y' + 12y = 0$ $y = e^{rt}$, $y' = re^{rt}$, $y'' = r^2 e^{rt}$
 $r^2 e^{rt} - 7re^{rt} + 12e^{rt} = (e^{rt})(r^2 - 7r + 12) = 0$ $(r-3)(r-4) = 0$ $r = 3, 4$
 $y(t) = c_1 e^{3t} + c_2 e^{4t}$

4. $y'' + 3y' + 2y = 0$ $r^2 + 3r + 2 = 0$ $(r+2)(r+1) = 0$ $r = -1, -2$
 $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

5. $y'' - 2y' - 25y = 0$ $r^2 - 2r - 25 = 0$
 $r = \frac{2 \pm \sqrt{4 + 100}}{2} = \frac{2 \pm \sqrt{104}}{2} = \frac{2 \pm 2\sqrt{26}}{2} = 1 \pm \sqrt{26}$

$y(t) = c_1 e^{(1+\sqrt{26})t} + c_2 e^{(1-\sqrt{26})t}$

6. $y'' + 2y' - 7y = 0$ $r^2 + 2r - 7 = 0$ $r = \frac{-2 \pm \sqrt{4 + 28}}{2} = \frac{-2 \pm \sqrt{32}}{2}$
 $y(t) = c_1 e^{(1+2\sqrt{2})t} + c_2 e^{(1-2\sqrt{2})t}$ $= \frac{-2 \pm 4\sqrt{2}}{2} = -1 \pm 2\sqrt{2}$

7. $y'' + y = 0$ $r^2 + 1 = 0$ $r = \pm i$
 $y(t) = c_1 \cos t + c_2 \sin t$

8. $4y'' - 9y = 0$ $4r^2 - 9 = 0$ $r = \pm \frac{3}{2}$
 $y(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{\frac{3}{2}t}$

9. $y'' - 8y' + 16y = 0$

$r^2 - 8r + 16 = 0 \quad (r-4)^2 = 0 \quad r = 4 \text{ repeated}$

$y(t) = c_1 e^{4t} + c_2 t e^{4t}$

10. $y'' + 2y' + 10y = 0 \quad r^2 + 2r + 10 = 0$

$r = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$

$y(t) = c_1 e^{-t} \cos 6t + c_2 e^{-t} \sin 6t$

11. $y'' - 5y' + 25y = 0 \quad r^2 - 5r + 25 = 0$

$r = \frac{5 \pm \sqrt{25 - 100}}{2} = \frac{5 \pm 5\sqrt{3}i}{2} = \frac{5}{2} \pm \frac{5\sqrt{3}}{2}i$

$y(t) = c_1 e^{\frac{5}{2}t} \cos(\frac{5\sqrt{3}}{2}t) + c_2 e^{\frac{5}{2}t} \sin(\frac{5\sqrt{3}}{2}t)$

12. $5y'' + 6y' + 8y = 0 \quad 5r^2 + 6r + 8 = 0$

$\frac{-6 \pm \sqrt{36 - 160}}{2(5)} = \frac{-6 \pm 2\sqrt{31}i}{10} = -\frac{3}{5} \pm \frac{\sqrt{31}}{5}i$

$y(t) = c_1 e^{-\frac{3}{5}t} \cos(\frac{\sqrt{31}}{5}t) + c_2 e^{-\frac{3}{5}t} \sin(\frac{\sqrt{31}}{5}t)$

13. $25y'' + 70y' + 49y = 0$

$25r^2 + 70r + 49 = 0$

$(5r+7)^2 = 0 \quad r = -\frac{7}{5} \text{ repeated}$

$y(t) = c_1 e^{-\frac{7}{5}t} + c_2 t e^{-\frac{7}{5}t}$

14. $y''' + y' = 0$

$r^3 + r = 0$

$r(r^2 + 1) = 0 \quad r = 0, \pm i$

$y(t) = c_1 + c_2 \cos t + c_3 \sin t$
 (e^{0t})

15. $y''' + 2y'' - y' - 2y = 0$

$r^3 + 2r^2 - r - 2 = 0 \quad r^2(r+2) - 1(r+2) = 0$

$(r^2-1)(r+2) = (r+1)(r-1)(r+2) = 0 \quad r = 1, -1, -2$

$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}$

16. $y^{IV} + y'' = 0 \Rightarrow r^4 + r^2 = 0 \quad r^2(r^2+1) = 0$
 $r = 0$ (repeated), $r = \pm i$

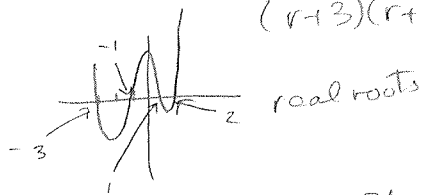
$y(t) = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$

17. $y^{IV} + 2y''' + y'' = 0 \Rightarrow r^4 + 2r^3 + r^2 = 0 \quad r^2(r^2+2r+1) = 0$
 $r = 0$ (repeated), $r = -1$ repeated

$y(t) = c_1 + c_2 t + c_3 e^{-t} + c_4 t e^{-t}$

18. $y^{IV} + y''' - 7y'' - y' + 6y = 0 \quad r^4 + r^3 - 7r^2 - r + 6 = 0$
 $(r+3)(r+1)(r-1)(r-2) = 0$

do this graphically



$y(t) = c_1 e^{-3t} + c_2 e^{-t} + c_3 e^t + c_4 e^{2t}$

19. $y''' - y = 0 \quad r^3 - 1 = 0 \quad (r-1)(r^2+r+1) = 0 \quad r = 1$

$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

$y(t) = c_1 e^t + c_2 e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) + c_3 e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t)$

20. $y^{IV} - y'' = 0 \quad r^4 - r^2 = 0 \quad r^2(r^2-1) = 0 \quad r = 0$ (repeated), $r = \pm 1$

$y(t) = c_1 + c_2 t + c_3 e^t + c_4 e^{-t}$

$$21. y^{VIII} + 8y^{IV} + 16y = 0$$

$$r^8 + 8r^4 + 16 = 0$$

$$(r^4 + 4)^2 = 0 \quad \text{all roots of } r^4 + 4 \text{ repeated}$$

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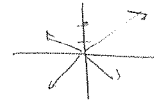
$$r^4 + 4 = 0 \Rightarrow r^2 = \pm 2i$$

$$r = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = 1+i$$

$$r = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = 1-i$$

$$r = \sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = -1+i$$

$$r = \sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = -1-i$$



$$y(t) = C_1 e^t \cos t + C_2 e^t \sin t + C_3 e^{-t} \cos t + C_4 e^{-t} \sin t + C_5 t e^t \cos t + C_6 t e^t \sin t + C_7 t e^{-t} \cos t + C_8 t e^{-t} \sin t$$

$$1. t^2 y'' + 5t y' + 4y = 0 \quad y(t) = t^n \quad y' = n t^{n-1} \quad y'' = n(n-1)t^{n-2}$$

$$(n^2 - n)t^n + 5n t^n + 4t^n = 0 \quad t^n [n^2 + 4n + 4] = 0 \quad n = -2 \text{ (repeated)}$$

$$y(t) = C_1 t^{-2} + C_2 t^{-2} \ln t$$

$$2. t^2 y'' + 4t y' + 2y = 0$$

$$n^2 - n + 4n + 2 = n^2 + 3n + 2 = 0$$

$$(n+2)(n+1) = 0 \quad n = -2, -1$$

$$y(t) = C_1 t^{-1} + C_2 t^{-2}$$

$$3. t^2 y'' - 4t y' + 4y = 0$$

$$n^2 - n - 4n + 4 = n^2 - 5n + 4 = 0$$

$$(n-4)(n-1) = 0 \quad n = 4, 1$$

$$y(t) = C_1 t + C_2 t^4$$

$$4. t^2 y'' - 4t y' + 6y = 0$$

$$n^2 - n - 4n + 6 = n^2 - 5n + 6 = 0 \quad (n-2)(n-3) = 0 \quad n=2,3$$

$$y(t) = c_1 t^2 + c_2 t^3$$

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$$5. t^2 y'' + 3t y' + y = 0 \quad (n^2 - n) + 3n + 1 = n^2 + 2n + 1 = 0$$

$$(n+1)^2 = 0 \quad n = -1 \text{ (repeated)}$$

$$y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t.$$