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Counting

This handout will cover three different situations in which we would like to count large numbers of items without physically listing out every possible case. The three situations are:

1) With Repetition and Where Order Matters: The Fundamental Counting Principle
2) Without Repetition and Where Order Matters: Permutations
3) Without Repetition and Where Order Doesn't Matter: Combinations

There is a fourth scenario we will not address here, and that is where repetition is allowed, but order doesn't matter (as in counting the number of ways you can choose a dozen donuts).

1. The Fundamental Counting Principle

The Fundamental Counting Principle described on this page begins with a basic property of counting. If we can break the problem up into smaller parts and count these smaller parts more easily, then we can multiply the number of the smaller parts together to discover the number of the whole.

The simplest example is with a two-digit number. One possible method of counting the number of two digit numbers would be to make a list of them all and then count them. But this is tedious and time consuming. Instead, what we can do is break up our two-digit number into two parts: the first digit and the second digit. For the sake of keeping track of these parts we can represent them as spaces or as boxes to hold their places.

______	______
digit 1	digit 2

Now what we need to determine is the number of possible values for each part. How many numbers can we put into the first digit? Well, we didn't specify that we are allowed to use numbers starting with zero so there is a little bit of ambiguity in the problem, but let's assume that we mean numbers that we would actually write as two digits normally. That means a number starting with 0 isn't possible, so we are left with *1*, *2*, *3*, *4*, *5*, *6*, *7*, *8* and *9* to fill the first digit. That's nine possibilities.

Now, how many numbers can we put in the second digit? Well, now we can use 0 because we do write numbers that end in zero as two digits. So our options for the second digit are all ten numbers *0*, *1*, *2*, *3*, *4*, *5*, *6*, *7*, *8* and *9*. So now our places look like this:

9	10
digit 1	digit 2

To get the number of two digit numbers, we just multiply 9*10 = 90. So there are 90 two-digit numbers. We know from experience that the smallest one is 10 and the biggest one is 99.
We can extend this principle to much more complicated situations, and I will refer back to it from time to time.

Worked Examples.
As the introduction to this handout states, the Fundamental Counting Principle is most often used whenever *Order Matters*. It can be used for cases that either allow or disallow Repetition, but I will concentrate my examples here on cases *With Repetition*, because the cases with No Repetition will be discussed in Part 3 of this handout.

Example 1. Suppose we wished to determine the number of three-letter words whose middle letter is a vowel and whose first and last letters were consonants. Right away we see that we have a three-letter word, so it makes sense to try to divide this problem up into three parts.

________	________	________
letter-1	letter-2	letter-3

Let's start with the middle letter. How many vowels are there in the alphabet? Well, there's *a*, *e*, *i*, *o* and *u*. If we were talking about real English words, we might also wish to count *y*; however, most problems of this type consider *y* a consonant. That gives us five possibilities for the middle letter. The rest of the alphabet is consonants, so since there are 26 total letters, 26-5 = 21 consonants. (Note: if you choose to do this problem counting *y* as a vowel, remember that it's also a consonant, however, this tends to make things too complicated because you'd have to eliminate things like yyy, etc.) This gives us the following:

___21___	___5___	___21____
letter-1	letter-2	letter-3

The total possible words of this configuration would then be 21*5*21 = 2205.

Example 2. How many possible telephone numbers are there for each telephone exchange?

First, we have to figure out what this problem is asking. The telephone exchange is the first three digits of a telephone number. For television shows, the exchange 555 has been set aside just for them--so they never have to use a real number. Another way of asking this question is: choose one telephone exchange (let's say yours), how many people can have the same first three digits as you?

Since a telephone has seven digits, we could start out this way:

________ ________ ________ ________ ________ ________ ________

We have seven slots, but since we are talking about fixing the first three digits, the exchange part of the number, we are really starting out with:

____1____ ____1____ ___1_____ ________ ________ ________ ________

Suppose I choose 555 as the exchange I'm working with, then there is only one possible choice for the first digit, and only one possible choice for the second, and one possible choice for the third digit. The other digits are what we still have to determine. This will be true for any fixed exchange.

The remaining four slots can be filled in with any number. There are no restrictions on the last four digits of a phone number. Phone companies like to give certain numbers to businesses and certain others to residents but that doesn't affect the total number of numbers. Each remaining digit can be any of: *0*, *1*, *2*, *3*, *4*, *5*, *6*, *7*, *8* and *9*. So, we'll fill in the remaining slots with 10's.

___1___ ___1___ ___1___ ___10___ ___10___ ___10___ ___10___

So the answer to our question is 1*1*1*10*10*10*10 = 10,000.

Example 3. How many alphanumeric 6-letter passwords are there?

By alphanumeric, we mean any letter or number. For each letter or space in the password we have ten number possibilities and 26 letter possibilities. Assuming the simplest scenario, where case doesn't matter and we aren't allowed to use special characters like # or what have you, each space in the password has 36 possible values. This gives us:

___36___ ___36___ ___36___ ___36___ ___36___ ___36___

The answer to our problem then is 36*36*36*36*36*36 or $(36)^6 =$ 2,176,782,336 possible combinations.

Problem Solving Tips.

What I try to do when solving these kinds of problems is look for a few key things before I begin, and as I go through the problem.

- Always, first read the problem all the way through. You want to find the question in the problem; that will tell you what exactly you're looking for.

- You want to find out how many parts you can divide the problem into. Sometimes the problem will be explicit, like examples A and C where we had a three-digit word or a six-letter password. Sometimes the problem will be less explicit and expect you to know something common about the word, as with Example 2 where we needed to know that telephone numbers are seven digits and the first three digits are the exchange. For younger students, this might not seem so obvious because so many cities are going to ten-digit phone numbers. However, we can do this problem for ten digits with a fixed area code in a similar fashion.

- You also want to look for any restrictions on any of the digits. In Example 2 we saw that we wanted to know only about each exchange or a single exchange, so we were fixing the first three digits of the number to a single possibility. In Example 1, we restricted the middle letter to just vowels and the outside letters to just consonants. Even though there are 26 possible letters, we aren't allowed to use all of them in each position. Other examples of this that we didn't use might be even/odd numbers, numbers less than a certain number, etc.

- Since you can also use this method for permutations, you also want to check to see whether or not repetition is allowed. (Am I allowed to use the same digit or letter twice? For instance is 99 or bib allowed? Check out the section on permutations to see examples of when it isn't allowed. With letters and numbers repetition is usually fine unless the problem specifies otherwise.)
- You also want to look for any possible ambiguities and attempt to resolve these logically. The problem may lend itself to one interpretation over another because of the context. Problems

which are worded properly will disambiguate, but we can't always rely on this in the real world. You may need to get additional information (if this is possible). If this isn't possible, chose the interpretation closest to everyday experience.

- ❖ Look for commonalities between problems. You'll often find that passwords will use the fundamental counting principle, but card games will use another method.

Practice Problems.
Find the number of items described in each situation.

1. How many five digit numbers are there?
2. How many five digit even numbers are there?
3. In Ohio, the regular license plates have three letters, followed by four numbers. How many different license plates are there?
4. Suppose you flip a coin five times? How many different sets of five flips are there?
5. Suppose you are going to a restaurant that serves items a la carte. You decide to order three things: one meat dish, one vegetable dish and one bread. You could order beef, chicken or fish for the meat dish. You could order carrots, beans, broccoli, or corn for the vegetable dish. And you could order biscuits, muffins, or sweet bread for the bread dish. How many different possible dinners could be served?
6. Suppose you are dealing four cards from a deck and you end up with one card from each suit. How many different hands of this kind are there? (Hint: remember that there are 13 cards in each suit.)
7. A couple is interested in having children. They've decided on having three. Suppose they've chosen the names Chris, Kelly and Pat (all unisex names). How many possible ways can they name their three children? (They won't give two children the same name.)
8. How many radio station call signs are there? (Hint: They have to start with W or K and have four letters.)
9. How many ways are there to roll three six-sided dice? (Pretend you are rolling one die three times.)
10. Suppose you have a keypad with twelve keys on it for your security system. How many different four-key passcodes are there?

Challenge Problems.

11. Suppose you have four friends and you want to buy tickets to the midnight showing of Star Wars: Episode VII. How many different ways can the five of you wait in line? (Hint: Once you pick one friend to stand first in line, they can't stand anywhere else in line. If there are five ways for the first person to stand in line, how many people are left to stand second? third? etc.?)
12. How many 8-digit passwords are there using alphanumeric characters and case matters? You may want to state your answer in scientific notation.
13. If it takes you five seconds to check each of the passwords in problem #12, how long will it take to test them all? State your answer in years.

Answers to Problems:
1. 10^5 = 100,000, if you allow 0 as a first digit, otherwise $9*10^4$=90,000. 2. 10^4*5 = 50,000. 3. 26^3*10^4 = 175,760,000. 4. 2^5 = 32. 5. 3*4*3 = 36. 6. 13^4 = 28,561. 7. 3*2*1 = 6. 8. $2*26^3$ = 35,152. 9. 6^3 = 216. 10. 12^4 = 20,736. 11. 5*4*3*2*1 = 120. 12. 62^8 = 2.18 x 10^{14}. 13. Divide your answer in 12 by (60*60*24*365) = 6,923,519 years.

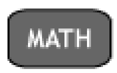
2. Permutations

In order to discuss permutations and combinations, we need to introduce a new mathematical notation !, used in constructing factorials. These are essential in our formulas for both permutations and combinations.

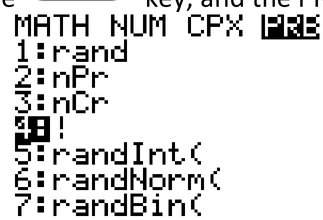
8! is defined this way: $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. To put this in words, to find the value of 8! Multiply all the numbers, starting with 8 and counting all the way down to 1 together. Factorials get big pretty quickly as you can see from this chart.

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

Note that 0!=1 not 0. This is because the identity in multiplication is 1 not 0.

You can find the factorial key under the 

key, and the PRB (Probability) menu:



Permutations involve all the possible arrangements (where order matters with no repetitions) of a fixed set of objects. Consider the following set of four numbers {*5, 6, 7, 8*}. All the possible permutations of this set of numbers are:

5678	*5687*	*5867*	*5876*	*6587*	*6578*	
5786	*5768*	*8756*	*8765*	*6758*	*6785*	*6857*
6875	*7568*	*7586*	*7658*	*7685*	*7865*	*7856*
8567	*8576*	*8657*	*8675*			

There are 24 possible permutations of these four objects. As the number of objects in the set grows, the number of possible permutations grows much more rapidly. If our set of objects has six elements in it, the number of possible permutations is 720.

Sometimes we wish to select fewer elements in the set than are present. For the same set above, if we choose two elements, the possible combinations are:

56	*57*	*58*	*65*	*67*	*68*	*75*
76	*78*	*85*	*86*	*87*		

There are twelve possible permutations. There is a formula we can use to find the number of possible permutations. It relates the total number of items in the set, *n*, with the total number of elements to be chosen, *r*. The formula is sometimes referred to with the label *nPr* or *P*(*n*, *r*), and is given by the relation $P(n,r) = nPr = \frac{n!}{(n-r)!}$ What this essentially tells us, is that the number of permutations is calculated by multiplying the number *n* by (*n-1*) by (*n-2*) and so on until you have the number *r* of terms you wish to count. [Remember that factorials, 4! for instance, just means $4 \cdot 3 \cdot 2 \cdot 1$.]

For instance, to calculate the number of permutations possible from a set of four elements when one chooses two elements, one multiplies $4 \cdot 3 = 12$. Equivalently we can use our expression above, $\frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 4 \cdot 3 = 12$

If one wishes to calculate the number of possible permutations from a set of four elements as we did above by listing, when one is choosing all four elements, one multiplies $4 \cdot 3 \cdot 2 \cdot 1 = 24$. Equivalently, we can use our expression above $\frac{4!}{(4-4)!} = \frac{4!}{0!}$. Remember that 0! is defined to be 1 so this is just 4! = $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Worked Examples.

Example 4. What are the number of possible permutations of a set of ten elements (using all ten)?

We can use our formula: *n* = 10 since that's the size of our set, and *r* = 10 since we are using all the elements. this gives us $\frac{10!}{(10-10)!} = \frac{10!}{0!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$.

Example 5. How many possible permutations are there of a two elements from a set of three elements, such as *abc*?

We can use our formula by determining the values for *n* and *r*. The size of our set is 3, and we want to use only two elements at a time. Therefore, $\frac{3!}{(3-2)!} = \frac{3!}{1!} = 3! = 3 \cdot 2 \cdot 1 = 6$.

Example 6. How many possible permutations are there of four elements from a set of ten elements, such as *0123456789*?

Here *n* = 10 and *r* = 3. $\frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$, which is exactly what we'd expect to get using the fundamental counting principle.

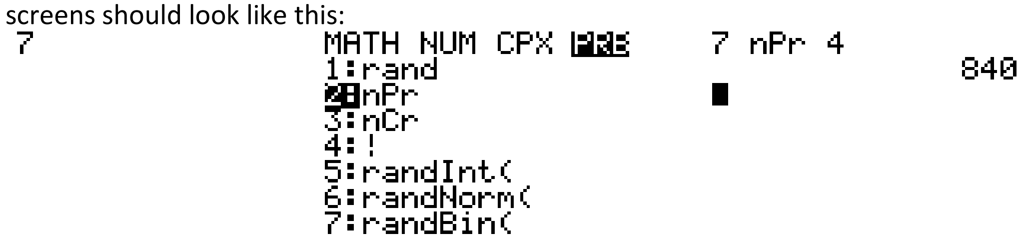
Problem Solving Tips
There are some things to look for in problems to tell if you are supposed to use permutations.

- Does order matter? If you answered yes, permutations may work.
- Is repetition allowed? If you answered yes, then permutations won't work.
- Sometimes your problem will say so explicitly, but usually it won't. If you are choosing balls in a large state lottery, each number can only be chosen once, and you win more if you get the order

correct as well, so permutations would be appropriate. Whenever you are ordering people since people can't be in two places at once, permutations are appropriate if order matters. For instance anytime you are assigning officers or first and second place. If you're unsure, put yourself into the problem... would you care if you were chosen first or second? If the answer is yes, then use permutations.

- Do you have a number of objects with the same label? Like blue balls or something? Then order is not really relevant, so you have to use something else.

- If you are using a TI-84 calculator, you can find factorial and permutations formulae preprogrammed into your calculator. Hit the MATH menu button. You have four menu options. Scroll over to the PRB menu. Factorial is command 4 under this menu as shown above and permutations as nPr is command 2. (You'll also see nCr; this is for combinations which we'll discuss below.) For both of these type a number first, *then* hit the factorial or permutation command. For factorial hit enter; for permutations, type the second number and then hit enter. Your screens should look like this:



Practice Problems.

1. How many different arrangements (permutations) are there of the digits *01234*?
2. What is the value of $P(7, 5)$?
3. Suppose that sixteen people show up at a theatre but inside the main doors there is only room for a queue of ten people between the doors and the ticket window. Find the number of such queues of ten people.
4. You are choosing a four-character password involving both letters and numbers (case of the letter is not important). Determine the number of possible four-character passwords that can be used without repeating any characters.
5. You are selecting five cards from a standard deck of playing cards (there are 52 cards in a standard deck). How many different ways can you choose those five cards?
6. In how many ways can 8 CD's be arranged on a shelf?
7. If a softball league has 12 teams, how many different end of the season rankings are possible? (Assume no ties).
8. In how many ways can the scrabble club of 20 members select a president, vice president and treasury, assuming that the same person cannot hold more than one office.
9. A key pad lock has 10 different digits, and a sequence of 5 different digits must be selected for the lock to open. How many key pad combinations are possible (suppose the keys remain pressed until the entire code is entered so that digits can't be used again)?

Challenge Problem.

10. How many 5-digit *even* numbers can you make from {1245789}?

Answers to Problems:
1. 120. 2. 2520. 3. 2.9×10^{10}. 4. 1,413,720. 5. 311,875,200. 6. 40,320. 7. 479,001,600. 8. 6840 9. 30,240. 10. 2520.

3. Combinations.

Combinations are similar to permutations except that now order doesn't matter. We are still selecting from a fixed set of elements, such as {*5, 6, 7, 8*}. The possible combinations we can choose, given no repetitions are:

5678

If we choose fewer than the maximum number of elements, such as two elements would be:

56 *57* *58* *67* *68* *78*

Like permutations, we have a formula for determining the number of possible combinations of elements without listing them out, but the formula is slightly more complicated. We can refer to the formula with the label *nCr*, *C*(*n*, *r*) or often just $\binom{n}{r}$. As with permutations, *n* represents the number of elements in the set, while *r* represents the number of elements to be chosen from the set. The formula for combinations is given by $\frac{n!}{r!(n-r)!}$.

For instance, in the first example above, we have a set of four elements, and we are selecting all four elements for our combination, so our formula gives us $\frac{4!}{4!(4-4)!} = \frac{4!}{4!(1)!} = 1$. In the second example, our formula gives us $\frac{4!}{2!(4-1)!} = \frac{4!}{2!2!} = \frac{4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1} = \frac{4\cdot3}{2} = \frac{12}{2} = 6$.

Combinations have a lot of applications in mathematics. We will be seeing some of them below when we discuss the binomial theorem. Another example is Pascal's triangle:

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1
Row 6	1 6 15 20 15 6 1

Each row of the triangle is the value of a given *n*, and the value in the triangle is the value of the combination *r+1* entries across the row. For example, in Row 6, the 20, is the 4th element in the row. If 4 = *r*+1, then *r* = 3. And if we use the formula given above, 20 is the value of *C*(6, 3). This method can be useful if you don't have a calculator handy, but we will be using a calculator or the formula above for most of our applications.

Worked Examples.

Example 7. How many dance couples can be formed from a group of eleven people?

We are choosing from a set of eleven, and we are choosing two elements, so $\binom{11}{2} = 11C2 = C(11,2) = \frac{11!}{2!(11-1)!} = \frac{11!}{2!(9)!} = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{11\cdot10}{2} = 55$. The combinations formula is the same as $\frac{P(n,r)}{r!}$.

Example 8. Find the value of $\binom{5}{3}$.

This is just a computational problem. Just plug into the formula. $\binom{5}{3} = 5C3 = C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = \frac{5\cdot4}{2} = 10$.

Example 9. Find the expansion of $(a+b)^4$.

This is a more complex example, but it also used combinations. We could multiply this out longhand, but according to the binomial theorem, each term of the expansion has the following shape: $\binom{n}{r} a^{n-r} b^r$. We begin with $r = 0$ and go up to $r = n$ where n is the degree of the exponent. The first term is $\binom{4}{0}a^4b^0 = a^4$. The second term is $\binom{4}{1}a^{4-1}b^1 = 4a^3b$. $\binom{4}{2}a^{4-2}b^2 = 6a^2b^2$. $\binom{4}{3}a^{4-3}b^3 = 4ab^3$. $\binom{4}{4}a^{4-4}b^4 = b^4$. Now add them all up to get $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Problem Solving Tips
Some things to look for when solving combination problems:

- Does the problem say specifically either no repetition or that order doesn't matter?

- If the problem is not explicit, combination problems are used when order or ranking is impossible or not of value. Consider people in a couple or a group; does it matter who gets picked for the couple first? People who are finalists in a contest; don't they all have the same status? Winners of a contest who all get the same prize. Problems with multiple objects with the same label, like blue balls and green balls. Card games: If you have a full house, does it matter which card you got first or last?

- For problems with people in it, try to put yourself in the problem and see if you care whether you are picked first or second. If you don't, then use combinations.

- Binomial problems use combinations... there are binomial probability problems that you may encounter in statistics that also uses this formula as well as the binomial theorem you might see in algebra.

Practice Problems.

1. How many ways can you be dealt a hand of five cards in poker from a standard deck of playing cards? (Remember that hands don't matter what order you get the cards in.)
2. Find $C(8, 6)$.
3. Find $C(8, 2)$.
4. A company is giving away a trip to Hawaii to its three top sellers. The company has 22 employees. How many possible ways could the vacation trip be given away?

5. One way of winning the lottery is getting all the numbers picked in any order. How many ways can a person pick numbers to cover all the possible combinations of winners if you have to pick six numbers out of 44?
6. In a conference of 9 schools, how many intraconference football games are played during the season if the teams all play each other exactly once?
7. Ms. Mitchell will choose 1 boy and 1 girl from her class to be the class representatives. If there are 4 boys and 7 girls in her class, how many different pairs of class representatives could she pick?

Challenge Problems.

8. There are 5 married couples and a group of three is to be formed out of them; how many arrangements are there if a husband and wife may not be in the same group?
9. A door can be opened only with a security code that consists of five buttons: 1, 2, 3, 4, 5. A code consists of pressing any one button, or any two, or any three, or any four, or all five. How many possible codes are there? (Assume all the codes need to be depressed simultaneously.)
10. A person has the following bills: $1, $5, $10, $20, $50. How many unique sums can one form using any number of these bills only once?
11. A three-person committee must be chosen from a group of 7 professors and 10 graduate students. If at least one of the people on the committee must be a professor, how many different groups of people could be chosen for the committee?
12. What are the chances of choosing three balls from a bowl filled with 2 red balls, 3 yellow balls, 4 green balls and 5 blue balls where the three balls you choose consist of exactly 1 green ball and 2 red balls.
13. Write out the expansion of $(4x - 3y^2)^6$.
14. What is the 4th term of the expansion of $(3x^3 - y)^8$.

Can you tell the difference?

In the problems below, you'll be asked to determine whether the problems are combinations, permutations or if they can only be done by the multiplication rule (fundamental counting principle), or some combination of these methods.

1. A baseball coach is choosing a line-up for his team…
2. A poker player wants to know how many different ways there are to get two pair…
3. How many passcodes are there if you can use both cases of the alphabet, any number, and any of 16 special characters?
4. A teacher is choosing three-player teams from her class of 21.
5. The number of ways you can choose 3 marbles from a jar if all the marbles are different.
6. The number of ways you can flip 10 quarters.
7. Special elections are held on a committee for officers…
8. The number of winning numbers in PowerBall…

Answers to Problems:
1. 2,598,960. 2. 28. 3. 28. 4. 1540. 5. 7,059,052. 6. 36. 7. 28. 8. 480. 9. 31 [Hint: add up all the individual cases.] 10. 31 [Hint: this one works exactly the same as #9.] 11. 840. 12. There are only four ways of choosing three balls in the manner described. The total number of ways of choosing balls in this problem is 14C3 = 364. Now divide the smaller by the larger. 4/364 = 0.01098911 or 1.099%. 13. $4096x^6 - 18432x^5y^2 + 34560x^4y^4 - 34560x^3y^6 + 19440x^2y^8 - 5832xy^{10} + 729y^{12}$. 14. $(8C3)(3x^3)^5(-y)^3 = 56*243x^5*(-y^3) = -13608x^5y^3$