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Discrete Dynamical Systems

Discrete dynamical systems are systems of variables that are changing over time measured in discrete units (rather than continuously) such as in days, weeks, seconds, etc. We will be looking at such systems that can be modeled linearly so that they can be modeled with a matrix. One common example is a predator-prey model where one species (the predator) survives by consuming the second species (prey), and the prey reproduces to replenish the species. The rate at which the prey is consumed by the predators can change the stability of the system. There are many examples of such systems that have only one variable, but we will be considering examples similar to the predator-prey model, or some chemical reaction models, where two (or more) variables are needed to describe the system.

*Side note: Markov chains that we looked at earlier in the course are also a type of discrete dynamical system. We will now be considering a slightly more generic situation where the entries of the matrix describing the system need not be probabilities.

We will consider the problem geometrical, and then discuss applications.

Consider a system of two (or more) variables that change over time according to a linear function according to the formula $A\overrightarrow{x_k} = \overrightarrow{x_{k+1}}$. A is the matrix of the linear transformation, and $\overrightarrow{x_k}$ is the state of the system at some time k, with $\overrightarrow{x_{k+1}}$ the state of the system at the next time step. A sequence of such time steps, $\overrightarrow{x_0}, \overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}, \overrightarrow{x_4}, \ldots \overrightarrow{x_n}$ is called a **trajectory**.

Example 1. Consider the system of equations given below relating two variables. Write the matrix that represents the system, and give ten points of the trajectory given an initial state vector of $\overrightarrow{x_0} = \begin{bmatrix} 5 \\ 20 \end{bmatrix}$. Plot the points.

$$\begin{aligned} .2a_k + .5b_k &= a_{k+1} \\ -.5a_k + 1.3b_k &= b_{k+1} \end{aligned}$$

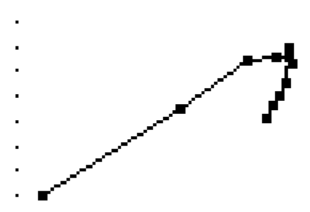
The matrix for the system, A, is equal to $\begin{bmatrix} .2 & .5 \\ -.5 & 1.3 \end{bmatrix}$. To find the next point in the trajectory, we just multiply the state vector by the matrix.

$$A\overrightarrow{x_0} = \begin{bmatrix} .2 & .5 \\ -.5 & 1.3 \end{bmatrix}\begin{bmatrix} 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 11 \\ 23.5 \end{bmatrix} = \overrightarrow{x_1}$$
$$A\overrightarrow{x_1} = \begin{bmatrix} .2 & .5 \\ -.5 & 1.3 \end{bmatrix}\begin{bmatrix} 11 \\ 23.5 \end{bmatrix} = \begin{bmatrix} 13.95 \\ 25.05 \end{bmatrix} = \overrightarrow{x_2}$$

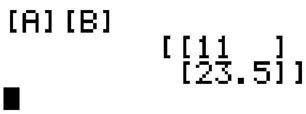
And so forth to obtain the sequence

$$\begin{bmatrix} 5 \\ 20 \end{bmatrix}, \begin{bmatrix} 11 \\ 23.5 \end{bmatrix}, \begin{bmatrix} 13.95 \\ 25.05 \end{bmatrix}, \begin{bmatrix} 15.315 \\ 25.59 \end{bmatrix}, \begin{bmatrix} 15.858 \\ 25.610 \end{bmatrix}, \begin{bmatrix} 15.976 \\ 25.363 \end{bmatrix}, \begin{bmatrix} 15.877 \\ 25.984 \end{bmatrix}, \begin{bmatrix} 15.667 \\ 24.541 \end{bmatrix}, \begin{bmatrix} 15.404 \\ 24.070 \end{bmatrix}, \begin{bmatrix} 15.116 \\ 23.588 \end{bmatrix}, \begin{bmatrix} 14.817 \\ 23.107 \end{bmatrix}$$

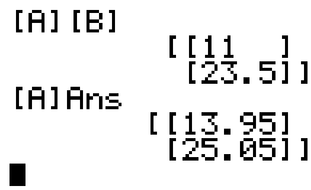
The plotted graph from the calculator is shown here, starting at the bottom left corner.

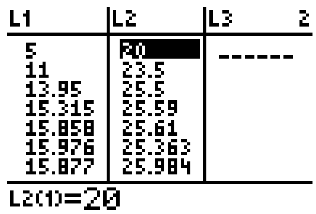


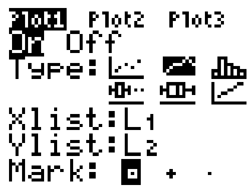
Calculator note: To obtain the trajectory with the fewest number of keystrokes, enter the matrix into Matrix [A] and the state vector into Matrix [B]. Then multiply [A]*[B] on the calculator screen as shown. This is the second state vector.



Then multiply the Matrix [A] by Ans as shown. The Ans key is [2nd] [(−)]. After that, just keep hitting enter to obtain the sequence of points. Then you can plot the points by going to the [STAT] menu and select Edit to enter the data into the lists. Use L1 and L2. To see the graph go to [2nd] [Y=] to turn on the Stat Plot. Choose Plot1. Select On, and the connected line graph (the second choice under Type. And Choose either the square box or the + sign for the graph. The dot will be impossible to see once the line is drawn. Then select [ZOOM] #9 (for ZoomStat) to see the graph; this function sets the window dimensions to best see the data in the lists. Be sure to delete any other graphing from your graphing screen or it will also plot and may appear on your graph. This procedure produced the graph shown above.







This model is interesting because it appears that both variables increase at first, and then after some time, both begin to decline. We are interested in the long-term behaviour of these systems, without, necessarily, plotting a hundred points of the trajectory. To study this, we will need to obtain the eigenvalues and eigenvectors of the matrix A.

Example 2. Find the eigenvalues and eigenvectors of the system in Example 1, and plot the eigenvectors on a graph. Compare with the established trajectory.

The matrix $A = \begin{bmatrix} .2 & .5 \\ -.5 & 1.3 \end{bmatrix}$. To find the eigenvalues of the matrix we subtract λ from the diagonal entries and calculate the determinant.

$$\begin{vmatrix} .2-\lambda & .5 \\ -.5 & 1.3-\lambda \end{vmatrix} = (.2-\lambda)(1.3-\lambda) + .25 = \lambda^2 - 1.5\lambda + .51 = 0$$

We can use the quadratic formula to find the eigenvalues from the characteristic equation.

$$\lambda = \frac{1.5 \pm \sqrt{(1.5)^2 - 4(1)(.51)}}{2(1)} = \frac{1.5 \pm \sqrt{.21}}{2} \approx 0.979\ldots, 0.52087\ldots$$

We have to be extremely careful finding the eigenvectors of the matrix since we have a square root involved. Rounding errors will likely make it almost impossible to solve and reduce the matrix in the calculator. But since there are only two variables, we know that both equations will be multiples of each other, so we can choose the one we would prefer to work with.

For the larger λ, we have:

$$\begin{bmatrix} .2-\lambda & .5 \\ -.5 & 1.3-\lambda \end{bmatrix} = \begin{bmatrix} .2-\frac{1.5+\sqrt{.21}}{2} & .5 \\ -.5 & 1.3-\frac{1.5+\sqrt{.21}}{2} \end{bmatrix} = \begin{bmatrix} -.55-\frac{\sqrt{.21}}{2} & .5 \\ -.5 & .55-\frac{\sqrt{.21}}{2} \end{bmatrix}$$

We'll use the bottom equation to avoid the radicals in the denominator.

$$-.5x_1 + \left(.55-\frac{\sqrt{.21}}{2}\right)x_2 = 0$$

$$x_1 = \frac{.55-\sqrt{.21}/2}{.5}x_2 \approx 0.642x_2$$

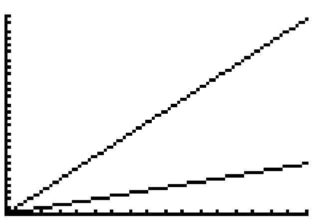
$$\overrightarrow{v_1} = \begin{bmatrix} 0.642 \\ 1 \end{bmatrix}$$

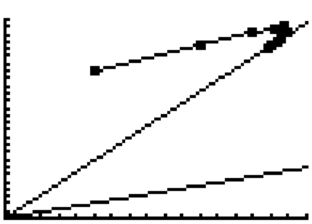
And we know that the other eigenvector, for the second eigenvalue is the conjugate of this:

$$x_1 = \frac{.55+\sqrt{.21}/2}{.5}x_2 \approx 2.496x_2$$

$$\overrightarrow{v_2} = \begin{bmatrix} 2.496 \\ 1 \end{bmatrix}$$

We can plot these as lines on our graph if we replace x_1 with x, and x_2 with y. The behaviour of the system is governed by the eigenvectors and the eigenvalues. We can see what's likely to happen from representing the state vectors in the coordinate system of the eigenvectors:





$$\overrightarrow{x_k} = c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}$$

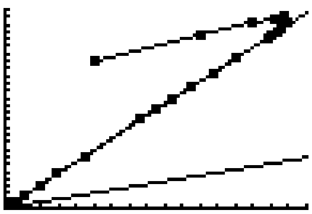
Each time we multiply the vector in this form by the matrix, we multiply each component eigenvector by the corresponding λ.

$$\overrightarrow{x_{k+1}} = c_1\lambda_1\overrightarrow{v_1} + c_2\lambda_2\overrightarrow{v_2}$$

So after n steps we have

$$\overrightarrow{x_{k+n}} = c_1\lambda_1^n\overrightarrow{v_1} + c_2\lambda_2^n\overrightarrow{v_2}$$

If |λ| > 1, then this component of the vector will grow without bound. If |λ|=1 (as we saw in the Markov chain examples) we will have an equilibrium vector that will remain stable over time (although if it's equal to negative one, it will oscillate between two values). If |λ| < 1, then raising this to a large power will make this component go to zero and it won't contribute much to the problem over time. When both eigenvalues are greater than one, the system will grow without bound. If one eigenvalue is greater than one, but the other one is less than one, the system will approach the eigenvector to the larger eigenvalue because the component from the other eigenvector will eventually vanish. If, as in this case, both eigenvalues are less than one, the long-term behaviour of the system is to collapse to zero. On the graph below, I plotted some more points of the trajectory for our system (spaced 5-10 steps apart).



Each of these types of systems have a special name. The one in our example, the origin acts like an **attractor** or a **sink** since both |λ|<1.

In the case where both |λ|>1, the origin acts like a **repeller** or a **source**.

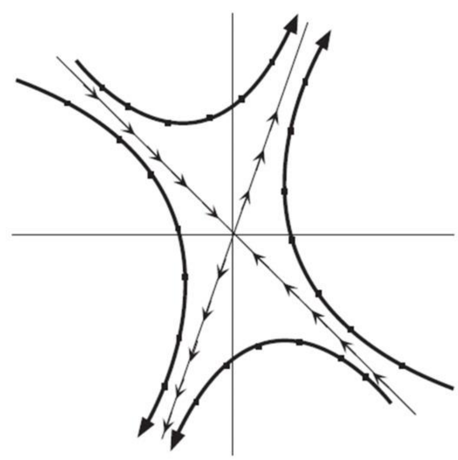
When one |λ|<1 and one |λ|>1, the origin is a **saddle point**, in that it attracts along one eigenvalue and repels along the other. The specific behaviour of the system is dependent on where the system begins, since some initial values will cause the system to collapse, but most will eventually move away from the origin in the direction of the eigenvector corresponding to the larger eigenvalue.

A graph of a saddle point solution is shown below with the behaviour of some sample trajectories.

The arrows on the eigenvectors indicate whether the origin is repelling or attracting along that vector.

In the special case of a Markov chain, one eigenvalue is equal to 1 and the other is less than one. This is what produces an **equilibrium** or *steady-state* value.

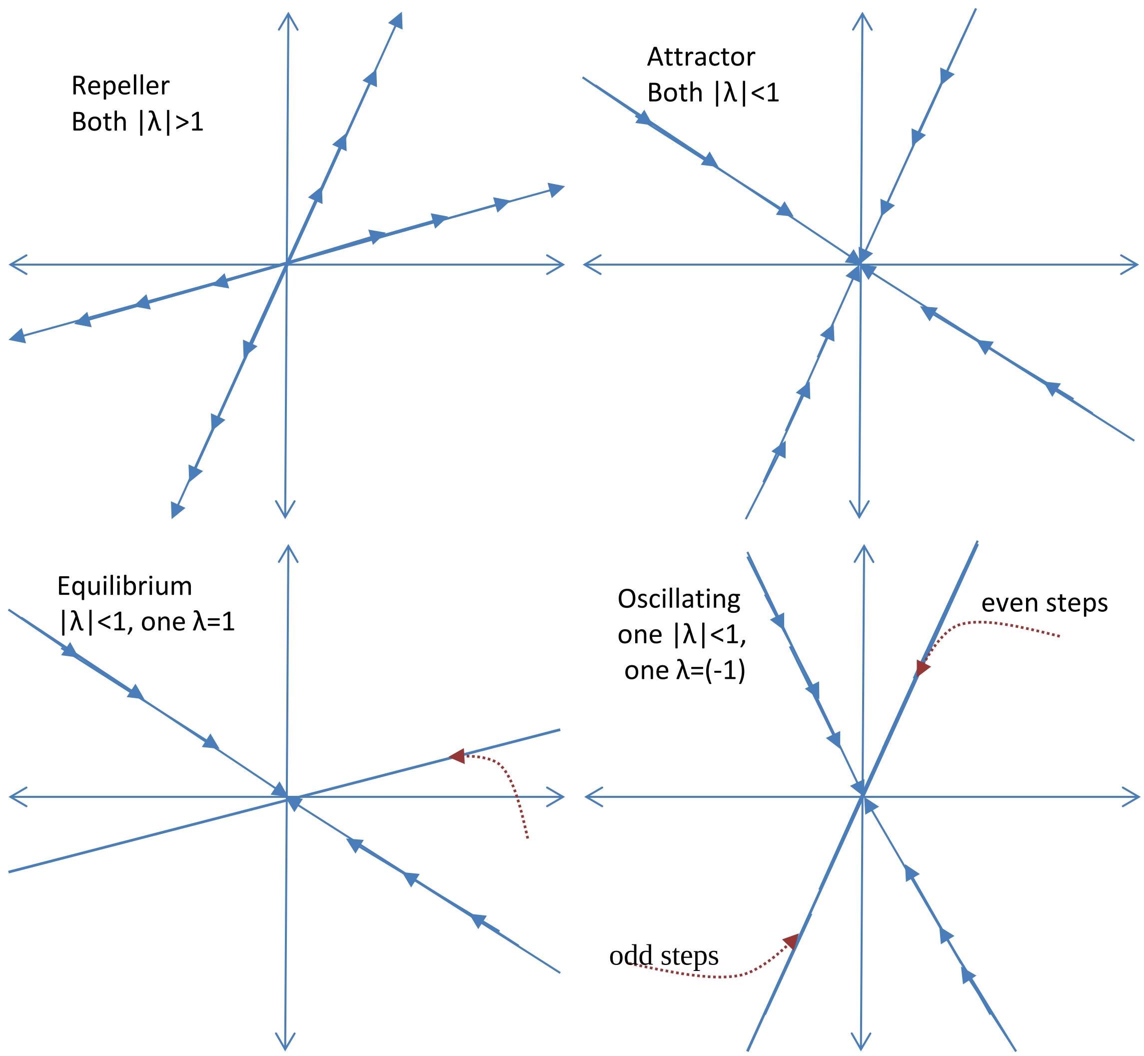
If one eigenvalue is equal to one and one is greater than one, the system will behave like a repeller unless you happen to get lucky and start with a point on the equilibrium vector.



Saddle

In the special case where one eigenvalue is -1, then the two cases above will be modified to include an **oscillation**. The equilibrium with be two valued and the system will flip back-and-forth between those two values, or it will behave like a repelling but with the same type of flipping behaviour relative to one of the eigenvectors, but the system will still move away from the origin.

These other types of cases with real eigenvalues are shown below. These types are graphs are called **phase portraits**.



The other kind of situation we could encounter is one where the eigenvalues of the matrix are complex.

Example 3. Consider the matrix of a discrete dynamical system given by $A = \begin{bmatrix} .2 & .5 \\ -.7 & 1.3 \end{bmatrix}$. This matrix is similar to the matrix we were examining before except for the entry on the bottom right corner. But this is going to make a big difference in the behaviour of the system when we find the eigenvalues.

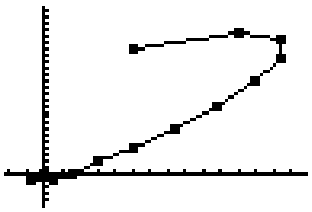
$$\begin{vmatrix} .2-\lambda & .5 \\ -.7 & 1.3-\lambda \end{vmatrix} = (.2-\lambda)(1.3-\lambda) + .35 = \lambda^2 - 1.5\lambda + .61 = 0$$

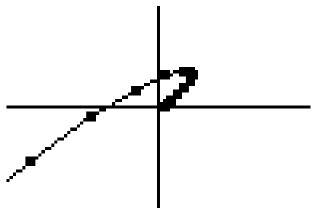
$$= \frac{1.5 \pm \sqrt{(1.5)^2 - 4(1)(.61)}}{2(1)} = \frac{1.5 \pm \sqrt{-.19}}{2} \approx 0.75 \pm .2179i$$

We no longer have real eigenvectors either, so we can't plot those, but we can examine a trajectory to see what happens to the system over time. Let's start with the same initial state that we used in Example 1, $\vec{x_0} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

$$\begin{bmatrix}5\\20\end{bmatrix}, \begin{bmatrix}11\\22.5\end{bmatrix}, \begin{bmatrix}13.45\\21.55\end{bmatrix}, \begin{bmatrix}13.465\\18.6\end{bmatrix}, \begin{bmatrix}11.993\\14.755\end{bmatrix}, \begin{bmatrix}9.776\\10.786\end{bmatrix}, \begin{bmatrix}7.348\\7.178\end{bmatrix}, \begin{bmatrix}5.059\\4.188\end{bmatrix}, \begin{bmatrix}3.106\\1.904\end{bmatrix}, \begin{bmatrix}1.573\\0.301\end{bmatrix}, \begin{bmatrix}0.465\\-.710\end{bmatrix}$$

The plot of these points and several more in the trajectory results in the graph below and blown up around the origin. The trajectory spirals into the origin rather than going to the origin more or less in a straight line. In Example 1, the coordinates of our system were never negative, but here, they not only dip into the negative, they pop back up into positive values again and on it goes with smaller and smaller numbers.





As with real eigenvalues, we can predict this kind of behaviour from the eigenvalues. We need to find the magnitude of the eigenvalues and to do that we need to remember how to find the modulus of a complex number.

$$\|a + bi\| = \sqrt{a^2 + b^2}$$

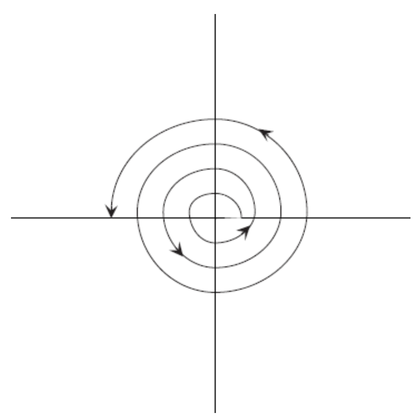
So in this example our $|\lambda| = \sqrt{(.75)^2 + \left(\frac{\sqrt{.19}}{2}\right)^2} = \sqrt{0.61} \approx 0.7810$. This value is clearly less than 1. When working with complex numbers, the magnitude of a complex number and its conjugate are exactly the same, so it's impossible to have a saddle point with a complex number. Except in the special cases where the magnitude is exactly 1, the origin will either be a repeller or an attractor.

Based on the complex eigenvalues, we can even figure out the rate of rotation. Recall the similarity transformation for complex eigenvalues with $C = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$. In our case, $C = \begin{bmatrix}0.75 & -.2179\\ 0.2179 & 0.75\end{bmatrix}$. Scaling this by the length (which we found above) we can find the angle. $\theta = cos^{-1}\left(\frac{0.75}{0.7810}\right), and\ \theta = sin^{-1}\left(\frac{0.2179}{0.7810}\right), so\ \theta \approx 0.283\ radians\ or\ 16.2°$.

These are both positive, so the angle is in the first quadrant. Be sure to check that your results to ensure that your answers are in the correct quadrant.

If $|\lambda|>1$, then the graph produced spirals outward. The speed at which this happens will depend on how close to 1 modulus is.

In the special case where λ has a modulus of exactly 1, in other words, it lies on the unit circle, the values produced by the system will remain on the unit circle. Whether the graph is strictly **periodic** or not will depend on whether the angle and the circle have a least common multiple.



Practice Problems.

1. For each of the matrices below, plot a trajectory of at least 15 points. Determine from the graph whether the system is an attractor, a repeller or a saddle point (or one of the special cases).
   a. $A = \begin{bmatrix}.38 & .24\\ -.36 & 1.22\end{bmatrix}, \vec{x_0} = \begin{bmatrix}15\\20\end{bmatrix}$.

b. $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{37}{21} & \frac{10}{21} \\ \frac{15}{21} & \frac{12}{21} \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

e. $A = \begin{bmatrix} 1 & 0.5 \\ 1 & 1.5 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

f. $A = \begin{bmatrix} 1.71 & -0.707 \\ 1 & 0 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$

g. $A = \begin{bmatrix} 1.8 & -.81 \\ 1 & 0 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} 15 \\ 3 \end{bmatrix}$

h. $A = \begin{bmatrix} 1.24 & -.97 \\ 1 & 0 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} -2 \\ 12 \end{bmatrix}$

i. $A = \begin{bmatrix} 1.24 & -1.03 \\ 1 & 0 \end{bmatrix}, \overrightarrow{x_0} = \begin{bmatrix} -5 \\ -8 \end{bmatrix}$

2. The word problems here are discrete dynamical systems. Build the matrix model of the system. Determine the behaviour of the origin in the model.
   a. The glucose and excess hormone concentration in your blood are modeled by the system of equations given by $\begin{cases} 0.978g_k - 0.006h_k = g_{k+1} \\ 0.004g_k + 0.992h_k = h_{k+1} \end{cases}$.
   b. The system of equations $\begin{cases} 2a_k = n_{k+1} \\ n_k + a_k = a_{k+1} \end{cases}$ models the birth of a lilac bush where *n* is the number of new branches, and *a* is the number of old branches.
   c. Two interacting populations of hares and foxes can be modeled by the system of equations $\begin{cases} 4h_k - 2f_k = h_{k+1} \\ h_k + f_k = f_{k+1} \end{cases}$.
   d. Suppose that spotted owls are dinning on tasty squirrels according to the model $\begin{cases} 0.4O_k + 0.3S_k = O_{k+1} \\ -0.325O_k + 1.2S_k = S_{k+1} \end{cases}$.