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Writing the Matrix of a Linear Transformation

Once we've established that a mapping is linear, it's often convenient to represent that transformation as a matrix. This can be a bit challenging when the basis for the space is infinite dimensional; however, we can consider the behaviour of the transformation on a finite subspace in order to construct a finite matrix for the transformation.

We will begin our discussion of linear transformation as close to the standard matrix form as possible, and then work our way toward more abstract spaces.

Consider the mapping $T: \vec{x} \in R^4 \mapsto T(\vec{x}) \in R^3$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_4 \\ x_1 + 2x_2 + x_3 \\ -x_2 + 5x_3 - 3x_4 \end{bmatrix}$. Write the matrix of the linear transformation.

We can verify quickly that this transformation is linear, as it will satisfy all the conditions of a linear transformation. So it can be written as a matrix. What will this matrix have to look like? Recall that a mapping from $R^n \mapsto R^m$ is an mxn matrix. So our matrix here will be a 3x4 matrix, and $T(\vec{x}) = A\vec{x}$.

To find the entries of A consider the generic 3x4 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$. If we multiply by the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, let's take one row of the matrix at a time.

Consider $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 2x_1 - x_4$. So for these expressions to be equal for any x_i, we need $a_{11} = 2, a_{12} = 0, a_{13} = 0, a_{14} = -1$.

Repeat this for each row of the matrix.

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = x_1 + 2x_2 + x_3$$

$$\begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = -x_2 + 5x_3 - 3x_4$$

Consequently, $a_{21} = 1, a_{22} = 2, a_{23} = 1, a_{24} = 0, a_{31} = 0, a_{32} = -1, a_{33} = 5, a_{34} = -3$.

So $A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & 5 & -3 \end{bmatrix}$.

You'll also notice that if each column represents the individual variables, the entries in each column are the coefficients of each variable on each line of the mapping.

Example 1. Find the matrix of the linear transformation $T: \vec{x} \in R^2 \mapsto T(\vec{x}) \in R^4$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 5x_2 \\ x_1 - 3x_2 \\ -x_2 \\ 9x_1 - 6x_2 \end{bmatrix}$.

This matrix will be a 4x2, and $A = \begin{bmatrix} 4 & 5 \\ 1 & -3 \\ 0 & -1 \\ 9 & -6 \end{bmatrix}$.

Example 2. Consider the polynomial in the standard basis $p(t) = 4 + 2t - t^2$. We can represent this polynomial in P_2 (since its highest degree is degree 2) as a vector in R^3 (since there are three components we need to account for (t^0, t^1, t^2). In this case, the vector would be $\vec{p} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ for some generic vector in P_2, and $\vec{p} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ for this polynomial specifically. Suppose we had a mapping $T: \vec{p} \in P_2 \mapsto T(\vec{x}) \in P_3$, where T is given by $T(\vec{p}) = (t-3)p(t)$. Find the matrix that represents T.

It helps here to perform the multiplication on a generic p(t) vector.

$$(t-3)p(t) = (t-3)(a_0 + a_1 t + a_2 t^2) = a_o t + a_1 t^2 + a_2 t^3 - 3a_0 - 3a_1 t - 3a_2 t^2$$

Collecting like terms we have:

$$(t-3)p(t) = (-3a_0) + (a_0 - 3a_1)t + (a_1 - 3a_2)t^3 + (a_2)t^3$$

Since this polynomial is now in P_3, we need 4 components to represent it as a vector in the way we did above. So $T(\vec{p}) = \begin{bmatrix} -3a_0 \\ a_0 - 3a_1 \\ a_1 - 3a_2 \\ a_2 \end{bmatrix}$.

Given that, we can represent it as a matrix in the same way we did in Example 1. $A = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 3. Suppose we have a function in the space defined by constant multiples of $\sin(t), \cos(t), \sin(2t), \cos(2t)$, which is to say all functions of the form $y(t) = a_1 \sin(t) + a_2 \cos(t) + a_3 \sin(2t) + a_4 \cos(2t)$. Since this is defined by 4 free variables a_i, this set of functions acts like a vector in R^4, i.e. $\vec{y} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$. Consider the linear transformation defined by the derivative operation. What is the derivative of y(t)? $T(y(t)) = y'(t) = a_1 \cos(t) - a_2 \sin(t) + 2a_3 \cos(2t) - 2a_4 \sin(2t)$, or if we put them in the same order as the functions in the original y(t) we get $T(y) = -a_2 \sin(t) + a_1 \cos(t) - 2a_4 \sin(2t) + 2a_3 \cos(2t)$. Recalling that the locations in the vector account for the trigonometric functions themselves, the new vector is now $T(\vec{y}) = \begin{bmatrix} -a_2 \\ a_1 \\ -2a_4 \\ 2a_3 \end{bmatrix}$. How can we represent this as a matrix? If $T(\vec{y}) = A\vec{y}$, then $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$. It's easy to show that if you multiply the original y(t) in vector form by the matrix, you do get the resulting vector representation of the derivative.

Example 4. Take the function defined above on the same space, and consider instead the antiderivative function (one where we insist that the constant of integration is zero). Here $\int y(t)\, dt = -a_1 \cos(t) + a_2 \sin(t) - \frac{1}{2} a_3 \cos(2t) + \frac{1}{2} a_4 \sin(2t)$. Reorganize as we did in Example 3, and then the vector $T(\vec{y}) = \begin{bmatrix} a_2 \\ -a_1 \\ \frac{1}{2} a_4 \\ -\frac{1}{2} a_3 \end{bmatrix}$. Thus the matrix of the linear transformation is $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$. You can check your answer by noting that if you take the derivative, and then integrate, you should get back to the same function, which is to say in matrix terms, you multiply by the identity. And it's easy to show that if you multiply the derivative matrix by the anti-derivative matrix, you get back the identity matrix.

In addition to these methods, you can also build a matrix of a linear transformation using a set of standard transformation matrices listed in our textbook. If you want to find the matrix that creates a series of such transformations, you can simply multiply the appropriate transformation matrices together.

For instance, we have such matrix transformations as:

a) Rotation matrix $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ for a counterclockwise rotation
b) Expansion or compression matrix: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ scales in the x_1 direction by a, and scales in the x_2 direction by b. (These scaling factors can be the same or different. If one of them is zero, then the matrix is called a projection matrix.)
c) Reflections: Across an axis, these look like the rotation matrix for $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Across the lines $x_1 = x_2$ or $x_1 = -x_2$ look like $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ respectively.

d) Shear matrices: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ shear in the x_1 direction and the x_2 direction respectively.

If we wanted to build a three dimensional situation where the transformation in two dimensions corresponds to one of these operations, we can use them as models in 2 of the three dimensions. For instance, if I wanted, in R^3 to rotate a vector by an angle θ in the x-z plane, and leave the y-direction unchanged, my matrix would look like $A = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$.

Example 5. Find a matrix that transforms a vector in R^2 to another vector in R^2 through a rotation of 30° clockwise, reflects over the line $x_1 = x_2$, and then stretches the x_2 component by a factor of 2.

The clockwise angle is a negative $\pi/6$, so the rotation matrix becomes: $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. The reflection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. And the scaling matrix for x_2 only is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Their product is: $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \sqrt{3} \\ \frac{\sqrt{3}}{2} & -1 \end{bmatrix}$.

Practice Problems.
For each of the linear transformations below, write the matrix of the linear transformation.

1. $T: \vec{x} \in R^3 \longmapsto T(\vec{x}) \in R^3$, where T is given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$.
2. $T: \vec{x} \in R^2 \longmapsto T(\vec{x}) \in R^3$, where T is given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 \\ x_1 + 4x_2 \\ x_2 \end{bmatrix}$.
3. $T: \vec{x} \in R^4 \longmapsto T(\vec{x}) \in R^4$, where T is given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{bmatrix}$.
4. Consider a polynomial in P_2 given by $p(t) = a_0 + a_1t + a_2t^2$. Define a linear operator T by $T(p(t)) = (2t^2 - t + 6)p(t)$ in P_4. Find the matrix of the transformation. [Hint: See Example 2.]
5. Consider a polynomial in P_2 given by $p(t) = a_0 + a_1t + a_2t^2$. Define a linear operator T by $T(p(t)) = (t^3 - 4)p(t)$ in P_5. Write the matrix of the transformation.
6. Consider a polynomial in P_3 given by $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$. Find the matrix of the linear transformation taking this vector into P_2 defined by the derivative operator $\frac{d}{dt}[p(t)]$. [Hint: See Example 3.]
7. Consider a polynomial in P_3 given by $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$. Find the matrix of the linear transformation taking this vector into P_4 defined by the antiderivative operator$\int_0^t (p(x)dx$. [Hint: See Example 4.]

8. Consider the function defined as $y(x) = a_1 e^x + a_2 e^{-x} + a_3 e^{5x} + a_4 e^{-7x}$. Write the matrix of the linear transformation defined by the derivative operator $\frac{d}{dx}[y(x)]$.
9. Consider a function defined as $y(x) = a_1 e^{3x}\cos(2x) + a_2 e^{3x}\sin(2x)$. Write the matrix of the linear transformation defined by the derivative operator $\frac{d}{dx}[y(x)]$.
10. For the same situation as in Problem #9, find the second derivative transformation matrix.
11. Find linear transformation matrix that transforms a vector in R^2 by rotating it counterclockwise by 225°.
12. Find a linear transformation matrix that transforms a vector in R^2 by first shearing it with factor k=3, and then reflecting it about the line $x_1 = -x_2$. [Hint: see Example.]
13. Find a linear transformation matrix that transforms a vector in R^3 by rotating it through an angle $2\pi/3$ in the x_2x_3-plane, then scales the x_1, x_2 directions by a factor of 4 and 2 respectively, and then reflects along the line $x_1 = x_3$.