Vector Spaces and Subspaces

A **vector space** is a set of elements that obeys the properties of vector addition and scalar multiplication that vectors also obey.

A **subspace** is a subset of elements in a vector space that *within that set* also obeys the properties of vector addition and scalar multiplication.

Do not get too hung, however, on the use of the term "vector". Vectors are the model object for this class of sets, but they are not the only things that create vector spaces. We will look at sets that are finite dimensional, which are isomorphic to traditional vector spaces in $\mathbb{R}^n$, but we will also look at infinite dimensional vector spaces which cannot be modeled as vectors, but still obey all the required properties.

Recall from our initial discussion of vectors what the properties of vector addition and scalar multiplication entail:

i. $\vec{u} + \vec{v}$ exists in the space
ii. $c\vec{u}$ exists in the space
iii. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
iv. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
v. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
vi. $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$
vii. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
viii. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
ix. $c(d\vec{u}) = (cd)\vec{u}$
x. $1\vec{u} = \vec{u}$

The good thing, though, is that we don't need to check each of these properties to guarantee they are all satisfied. As it turns out, these properties depend on three basic properties that all the rest follow from:

i. $\vec{0}$ exists in the space
ii. $\vec{u} + \vec{v}$ exists in the space, where each $\vec{u}, \vec{v}$ are in the space
iii. $c\vec{u}$ exists in the space, where $\vec{u}$ is in the space, and c is a real scalar

(We can also talk about complex vector spaces, but we will consider only real vector spaces in this handout.)

Property ii is sometimes called "closed under addition", and property iii is sometimes referred to as "closed under scalar multiplication".

These properties may remind you of linear transformations. That's because our linear transformation properties move from element from one vector space into another element in another. What our linear transformation properties guarantee is that the target space is a subspace. Therefore, these properties are related but they are not identical. Vector spaces are *sets* of elements while the linear transformation properties refer to the *mapping* from one set into another.

Example 1.

Define $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a + b + c = 0 \right\}$.

The elements of W are a subset of $\mathbb{R}^3$. What we want to know is if the set of vectors defined this way forms a subspace of $\mathbb{R}^3$, i.e. do the elements in W satisfy the properties of a vector space. We determine this by checking the three required properties. All of them must be obeyed to be a subspace.

i. Check the first condition, is $\vec{0}$ in the set? Yes, since we can let a,b,c=0. It holds that a+b+c=0+0+0=0 so this element is in the set, and this does give us $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the $\vec{0}$ element in $\mathbb{R}^3$.

ii. What if we add two elements in the set W? Consider two elements $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *and* $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ in W such that a+b+c=0 and d+e+f=0. Adding them we obtain the vector $\begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$.
Does (a+d) +(b+e)+(c+f)=0? Yes, since we can rearrange: (a+b+c)+(d+e+f)=0+0=0. So the sum is in W.

iii. What if we multiply a vector in W by a real scalar? Consider $k \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$. To be in W, ka+kb+kc must equal 0, and it does since ka+kb+kc=k(a+b+c)=k(0)=0.

The set W satisfies all the conditions, and so it is a subspace of $\mathbb{R}^3$.

Example 2.

Consider the set $H = \left\{ \begin{bmatrix} a \\ b^2 \end{bmatrix}, a, b \; real \right\}$ as a subset of $\mathbb{R}^2$. Is H a subspace?

If H is not a subspace, we need only to show that one of the conditions fails. To prove that it is a subspace, we must show that they all work. If you're not sure, proceed methodically and stop if anything doesn't work out, but think carefully before you assume the condition holds. If you have an intuition about a set, you can go directly to the disproof. You need only one example to fail. Here, it may help to note that H can be defined another way (and still have exactly the same elements) as $H = \left\{\begin{bmatrix} x \\ y \end{bmatrix}, x\ real, y \geq 0\right\}$. The $y \geq 0$ condition comes from the fact that because b was real in the other definition, all squares of real numbers are positive or zero, so this is the same condition stated another way. This definition, however, is a little easier to see how one of the conditions will fail.

iii. Scalar multiplication will fail if we consider the element $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, which is in H and multiply it by the real scalar (-1). $(-1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$, but this is not in the set since the second term is not positive, or put in terms of the original definition, there is no real b such that $b^2 = -4$.

We must conclude, therefore, that H is not a subspace.

In addition to traditional vectors forming vector spaces, we can also talk about any other sets that are isomorphic to $\mathbb{R}^n$ or a subspace of $\mathbb{R}^n$ as long as they add term-by-term, and scale term-by-term. Examples of this would include matrices such as $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which we can treat as isomorphic to $\vec{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ in $\mathbb{R}^4$, or polynomials (say, of degree less than or equal to three) such as $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ which is isomorphic to $\vec{v} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ also in $\mathbb{R}^4$. Any set isomorphic to a vector space will itself be a vector space. Whether a subset of these sets is a subspace will depend on the same tests we've previous considered.

Example 3.

Consider the set of 2x2 matrices with $M = \left\{\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, a, b, c\ real\right\}$. Is this a subspace (of all 2x2 matrices)?

i. If a,b,c=0, then $\vec{W} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is the $\vec{0}$ vector in the space, and it is in the set.

ii. Consider $\vec{U} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}, and\ \vec{V} = \begin{bmatrix} d & e \\ f & 0 \end{bmatrix}$. The sum of these is $\vec{U} + \vec{V} = \vec{A} = \begin{bmatrix} a+d & b+e \\ c+f & 0 \end{bmatrix}$. This entry is in the space, since the a_{11}, a_{12}, a_{21} are all real, and the a_{22} entry is 0. So the space is closed under addition.

iii. Consider $k\vec{U} = k\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & 0 \end{bmatrix}$, which also fits the required pattern of the set, since ka, kb, and kc are all real, and the required second row-second column position is zero. So the set is closed under scalar multiplication.

Since the set meets all the requirements, M is a subspace.

Example 4.
Is the set of all polynomials of degree less than or equal to three, called $\mathbb{P}_3$, a subspace of $\mathbb{P}_n$ (the set of all polynomials of degree less than or equal to n)? These are polynomials of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$.*

i. Is the $\vec{0}$ vector in the set? Well, the zero vector for the set of polynomials is p(t)=0. We can obtain that by allowing all the coefficients, a_i be equal to zero.
ii. Is the set closed under addition? It should seem clear that if you add two polynomials with highest degree no larger than three, you will obtain another object of the same kind, but we can check the description explicitly: $p(t) = a + bt + ct^2 + dt^3, q(t) = e + ft + gt^2 + ht^3$. Adding, we get $p(t) + q(t) = (a + bt + ct^2 + dt^3) + (e + ft + gt^2 + ht^3) = (a + e) + (b + f)t + (c + g)t^2 + (d + h)t^3$. Are each of the coefficients still real? Yes, so as we expected, it is closed under addition.
iii. Finally, checking scalar multiplication. What do you expect? If you multiply a polynomial of a certain degree by a real scalar, do you change the degree, or do you get another polynomial of the same degree? Check the math: $kq(t) = k(e + ft + gt^2 + ht^3) = (ke) + (kf)t + (kg)t^2 + (kh)t^3$. Are all the coefficients still real? As expected, they are, so it is closed under scalar multiplication.

*Note: This is a different set than polynomials of exactly degree three, since in such a set $a_3 \neq 0$. The wording of the set descriptions can matter a great deal, since the set of polynomials of degree exactly three does not contain the zero vector, since the polynomial $p(t) = 0$ has no degree formally speaking (whether that's less than three is one thing since most constants are considered to be degree zero). What is clear, however, is that this is not of degree exactly three.

One can also note as we did previously that $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$, so also for that reason, we conclude the $\mathbb{P}_3$ is a subspace.

In addition to finite dimensional vector spaces as in our first four examples, we can also consider even more abstract vector spaces, particularly those with infinite dimensions.

Example 5.
Consider the set N of all functions that are neither even nor odd. Is this a subspace of the set of all functions?

Recall that even functions are functions such that $f(x) = f(-x)$ like $x^2, |x|, 5 + x^4, \cos(x)$, etc. and odd functions are functions such that $f(-x) = -f(x)$ like $x, x^3, \frac{1}{x}, \sin(x)$, etc. Functions which are neither odd nor even may be composed of combinations of odd and even terms, or may contain terms like $e^x, \sqrt{x}, \ln(x)$, etc., functions which are neither odd nor even on their own.

As is turns out, this is not a subspace, for two reasons, though, of course, you only need to show that one of the conditions fails.

i. Does N contain the zero function? In other words, is the function f(x)=0 even, odd or neither? Zero, like other constant functions is an even function, and so therefore, f(x)=0 is not in the set N, and so it cannot be a subspace.
ii. N also fails the addition test. Consider the functions $f(x) = 3 - x$ and $g(x) = x + x^2$. These functions are in the set N since they contain both odd and even elements, but if you add them $f(x) + g(x) = 3 - x + x + x^2 = 3 + x^2$, which is an even function, and so the sum is no longer in the set.

So N is not a subspace.

Example 6.
Consider the set F with elements of all infinite series with a finite sum, i.e. $\sum_{i=0}^{\infty} a_i = L$. Is this a subspace of the set of all infinite series?

i. Is the zero element in the set? Sure it is, since there is a finite sum such that each $a_i = 0$ for all I, and the sum of that $\sum_{i=0}^{\infty} 0 = L = 0$. We can add this series to any other series and obtain the second series back again, so it does act like the additive identity should.
ii. Do two elements in the set add to another element in the set? In other words, if I add two series with finite sums together, do I obtain a series with a finite sum? Sure, since $\sum_{i=0}^{\infty} a_i + \sum_{i=0}^{\infty} b_i = \sum_{i=0}^{\infty}(a_i + b_i)$, and if the sum of $\sum_{i=0}^{\infty} a_i = L$, and the sum of $\sum_{i=0}^{\infty} b_i = M$ then the sum of $\sum_{i=0}^{\infty}(a_i + b_i) = L + M$ which must also be finite if L and M are finite. So the sum is in the set.
iii. Are scalar multiples in the set? Yes, since $k\sum_{i=0}^{\infty} a_i = kL$, and if k is a real scalar, then it is finite, and the product of finite numbers is also finite.

Therefore, F is a subspace.

The main error that students make with these problems is to go through the motions of checking the three conditions, but then to not think about them. You must think about what they mean, particularly for the more abstract cases, and not get too hung up on the notation.

Practice Problems.
Below is a list of sets with generally increasing levels of abstraction. Determine if each set is a vector space (subspace). Be sure to check all three conditions if it is a subspace. If it is not a subspace, prove that it is not by showing how at least one of the conditions is violated by elements in the set.

1. $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a = b + c \right\}$

2. $X = \left\{ \vec{b} \in R^3 | where\ A\vec{x} = \vec{b}, A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 2 & 1 & -1 \\ 0 & 3 & -2 & 0 \end{bmatrix}, \vec{x} \in R^4 \right\}$ [Hint: consider a random element in $\mathbb{R}^4$ like $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$, apply the transformation, and check the properties for a subspace. What can you say in general about a linear transformation from a vector space into another set?]
3. $Q = \left\{ \vec{b} \in R^4 | where\ T(\vec{x}) = \vec{b}, T(\vec{x}) = \begin{bmatrix} 3x_1 - 2x_4 \\ 6x_2 \\ -1 \\ x_3 - x_4 \end{bmatrix}, \vec{x} \in R^4 \right\}$ [Hint: Compare with #2.]
4. $Y = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, xy \geq 0 \right\}$
5. $Z = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x^2 + y^2 \geq 1 \right\}$
6. $S = \left\{ \begin{bmatrix} a & b & c \\ -b & a & d \end{bmatrix}, a + b + c = d \right\}$
7. $T = \left\{ \begin{bmatrix} a & 2 \\ 0 & b \end{bmatrix}, a, b\ real \right\}$
8. $U = \left\{ \begin{bmatrix} a \\ b^3 \end{bmatrix}, a, b\ real \right\}$
9. $B = \left\{ \begin{bmatrix} a + 2 & b \\ c & d \end{bmatrix}, a + b + c = d \right\}$
10. $\mathbb{C}$ = $\{the\ set\ of\ all\ complex\ numbers\ of\ the\ form\ a + bi, where\ a, b\ are\ real\}$
11. $D = \{set\ of\ all\ polynomials\ of\ the\ form\ p(t) = at^2, a\ real\}$
12. $A = \{set\ of\ all\ polynomials\ of\ degree\ exactly\ 2\}$ [Hint: note the difference in definition between D and A. What is being excluded in A?]
13. $G = \{the\ set\ of\ all\ polynomials\ of\ degree\ less\ than\ five\ but\ greater\ than\ one\}$
14. $J = \{the\ set\ of\ all\ polynomials\ such\ that\ p(t) divides\ evenly\ by\ (t - 1)\}$ [Hint: write p(t) in factored form, with the factor (t-1) pulled out. What does the other factor look like?]
15. $K = \{the\ set\ of\ all\ functions\ with\ an\ x - intercept\ at\ x = 3\}$
16. $L = \{the\ set\ of\ all\ functions\ with\ a\ y - intercept\ at\ 0\}$ [Hint: in other words, f(0)=0.]
17. $P = \{the\ set\ of\ all\ functions\ such\ that\ f(0) = 2\}$
18. $R = \{the\ set\ of\ all\ convergent\ definite\ integrals\}$
19. $E = \{the\ set\ of\ all\ even\ functions: f(x) = f(-x)\}$
20. $0 = \{the\ set\ of\ all\ odd\ functions: f(-x) = -f(x)\}$
21. $\Delta = \left\{ the\ set\ of\ all\ functions\ of\ the\ form\ f(x) = ax^{-n},\ where\ n\ is\ any\ natural\ number,\ ie. f(x) = \frac{a}{x^n} \right\}$
22. $\Omega = \{the\ set\ of\ all\ functions\ with\ a\ vertical\ asymptote\ at\ x = -2\}$
23. $\Gamma = \{the\ set\ of\ all\ functions\ defined\ on\ the\ interval\ [0,1]\}$