Inner Products

An inner product, in its most general form, is an extension of the dot product of two vectors that results in a numerical value. For this reason, it is also sometimes called a scalar product.

Notation for the dot product on vectors $\vec{u}$ and $\vec{v}$: $\vec{u}\cdot\vec{v}$ and is calculated by $\vec{u}^T\vec{v}$.

Sometimes the general notation for a vector product uses angle bracket notation $\langle\vec{u},\vec{v}\rangle$, although this notation is more common with inner products defined over infinite vector spaces composed of functions than with true vectors.

But what does the inner product mean when we are working on functions? We can define it any way we like really, as long as the result is a scalar. One common way to define an inner product on functions is $\langle f,g\rangle = \int_{-a}^{a} f(x)g(x)dx$, with some particular version of a. Commonly, a=1, but it can depend on the application. In Quantum Mechanics, the inner product is over a set of complex functions, so they use $\langle f,g\rangle = \int_{-\infty}^{\infty} \overline{f(x)}g(x)dx$, by taking the complex conjugate of the first function.

To be a valid inner product, the following conditions must hold:

1) $\langle\vec{u}+\vec{v},\vec{w}\rangle = \langle\vec{u},\vec{w}\rangle + \langle\vec{v},\vec{w}\rangle$
2) $\langle k\vec{u},\vec{v}\rangle = k\langle\vec{u},\vec{v}\rangle$
3) $\langle\vec{u},\vec{v}\rangle = \overline{\langle\vec{v},\vec{u}\rangle}$ (in a real vector space, the conjugation does nothing)
4) $\langle\vec{u},\vec{u}\rangle \geq 0$ and =0 only when $\vec{u}=\vec{0}$

Inner products are used in the definition of orthogonality. In the case of normal vectors, a dot product of zero translates into the geometric relationship of perpendicularity. However, in the more general case, orthogonal vectors or functions are simply those vectors or functions whose inner product is zero. Working with such orthogonal functions can make a host of calculations easier.

Example 1. Calculate the inner product of $\begin{bmatrix}-1\\2\\0\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\\5\\0\end{bmatrix}$. Determine if the vectors are orthogonal.

Transposing a vector changes it from a column vector into a row vector:

$$\begin{bmatrix}-1 & 2 & 0 & 3\end{bmatrix}\begin{bmatrix}2\\1\\5\\0\end{bmatrix} = (-1)(2)+(2)(1)+(0)(5)+3(0) = 0$$

Since the dot product is zero, the vectors are orthogonal.

Example 2. Given the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$ on the functions $f(x)=1,\ g(x)=1-3x^2$. Determine if they are orthogonal.

$$\int_{-1}^{1} 1(1-3x^2)dx = \int_{-1}^{1} 1-3x^2dx = x-x^3|_{-1}^{1} = 1-1-1+1 = 0$$

They are orthogonal since the inner product is zero.

Polynomials of a given size can be represented by vectors as well. If a given polynomial in $\mathbb{P}_2$ is given by $p(x) = a_0 + a_1x + a_2x^2$, then we can represent the vector by $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Then we can calculate the vector inner product. Orthogonality often depends both on the representation and the inner product used. The two polynomials which were orthogonal under the inner product in Example 2, are not orthogonal if we use the vector representation.

$$f(x) = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, g(x) = 1 - 3x^2 \to \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$
$$\vec{u}^T\vec{v} = [1 \quad 0 \quad 0]\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = (1)(1) + (0)(0) + (0)(-3) = 1 \neq 0$$

Example 3. Construct a vector orthogonal to $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$.

We can do this using the inner product. Consider the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The inner product with our given vector is $[1 \quad -2 \quad 4]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a - 2b + 4c = 0$. To be orthogonal, it must be equal to zero. This is the only equation we have to satisfy, so we have two free variables. Select values for b and c, and the equation will force a to be something that will make things work. Suppose I choose b=1 and c=1, then a-2b+4c=0 becomes a-2+4=0 or a+2=0. So a must equal -2. Thus our orthogonal vector is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Check the dot product again to see.

We can construct multiple vectors this way using these equations as a system. Each vector we add to the system restricts our free variables until we have a complete set. We can do the same thing with inner products on functions.

Example 4. Suppose that you are given the set of polynomials $\{1, x, 1 - 3x^2\}$ as a set of vectors that are orthogonal to each other. Find a 4th polynomial (of degree 3) that is orthogonal to all of these using the inner product in Example 2.

We could check that the orthogonality condition is satisfied for all combinations of these vectors, but I leave that as an exercise for the reader. We are going to construct a set of three equations based on taking the inner product of the new polynomial with the three we are given. Let's call the new polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Our three inner product calculations are as follows:

$$\langle f, g \rangle = \int_{-1}^{1} 1(a_0 + a_1x + a_2x^2 + a_3x^3)dx = \int_{-1}^{1} a_0 + a_2x^2\, dx + \int_{-1}^{1} a_1x + a_3x^3 dx$$

This second integral is an odd function, and so equals zero. That leaves us with

$$2\int_0^1 a_0 + a_2x^2\,dx = 2a_0x + \frac{2a_2x^3}{3}\bigg|_0^1 = 2a_0 + \frac{2a_2}{3} = 0$$

Repeating this procedure for the others.

$$\langle f,g\rangle = \int_{-1}^1 x(a_0 + a_1x + a_2x^2 + a_3x^3)dx = \int_{-1}^1 (a_0x + a_1x^2 + a_2x^3 + a_3x^4)dx =$$

$$\int_{-1}^1 a_1x^2 + a_3x^4dx + \int_{-1}^1 a_0x + a_2x^3 = 2\int_0^1 a_1x^2 + a_3x^4dx$$

$$\frac{2a_1x^3}{3} + \frac{2a_3x^5}{5}\bigg|_0^1 \frac{2a_1}{3} + \frac{2a_3}{5} = 0$$

$$\langle f,g\rangle = \int_{-1}^1 (1-3x^2)(a_0 + a_1x + a_2x^2 + a_3x^3)dx$$

$$= \int_{-1}^1 (a_0 + a_1x + a_2x^2 + a_3x^3 - 3a_0x^2 - 3a_1x^3 - 3a_2x^4 - 3a_3x^5)dx =$$

$$\int_{-1}^1 (a_0 + a_2x^2 - 3a_0x^2 - 3a_2x^4)dx + \int_{-1}^1 (a_1x + a_3x^3 - 3a_1x^3 - 3a_3x^5)dx =$$

$$2\int_0^1 (a_0 + a_2x^2 - 3a_0x^2 - 3a_2x^4)dx$$

$$2\left[a_0x + \frac{a_2x^3}{3} - \frac{3a_0x^3}{3} - \frac{3a_2x^5}{5}\right|_0^1 = 2a_0 + \frac{2a_2}{3} - 2a_0 - \frac{6a_2}{5} = -\frac{8a_2}{15} = 0$$

This provides us with three conditions: $-\frac{8a_2}{15} = 0$, $\frac{2a_1}{3} + \frac{2a_3}{5} = 0$, and $2a_0 + \frac{2a_2}{3} = 0$. Since the first condition says that $a_2 = 0$, then a_0 must also equal zero based on the last condition. That leaves the middle condition providing one free variable. In order to get a whole number, I'm going to solve this equation for a_1 and choose a value that cancels with the denominator.

$$\frac{2a_1}{3} + \frac{2a_3}{5} = 0 \rightarrow 10a_1 = -6a_3 \rightarrow a_1 = -\frac{3a_3}{5}$$

If $a_3 = 5$, then $a_1 = -3$. This makes our polynomial $p(x) = -3x + 5x^3$. You can check this to see that indeed, all the polynomial produce a zero inner product with each other.

Practice Problems.
Use the indicated inner product to determine if the vectors or functions are orthogonal.

a. Use the standard vector dot product.
   1. $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$
   2. $\left\{\begin{bmatrix}-1\\2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-3\\0\\5\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-1\\0\end{bmatrix}\right\}$

3. Find two more vectors orthogonal to $\begin{bmatrix} 1 \\ -3 \\ 5 \\ 4 \end{bmatrix}$

b. Use the inner product $\langle f,g \rangle = \int_{-a}^{a} f(x)g(x)dx$ with the indicated value of a to determine if the functions are orthogonal.

4. f(x)=cos(mx), g(x)=cos(nx) with m≠n. [Hint: You'll need a trig identity for this.] Assume m,n are integers, and a=π.
5. $\{1, x, 1-3x^2, -3x+5x^3\}$, with a=1
6. Follow Example 4 to find a 4th degree polynomial orthogonal to the polynomials in problem 5.

c. Use the inner product $\langle f,g \rangle = \int_{-a}^{a} \overline{f(x)}g(x)dx$ with the indicated value of a to determine if the functions are orthogonal.

7. $f(x) = xe^{ix^2}, g(x) = xe^{-ix^2}$ with a=∞. You can use Wolfram Alpha to complete the integration.