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Writing Proofs

One of the most important components of doing advanced mathematics is doing proofs. Proofs are more than just mere calculations; rather they tend to combine calculations with justification. One of the most difficult aspects for students first learning proofs is that they've become used to mere symbolic manipulation and doing proofs requires reintroducing words back into the process. Which steps require justification depends on the audience you are writing for. "Obvious" steps don't need justification, as long as they are part of a knowledge base you are certain is shared with your audience, but appealing to theorems or definitions will need to be employed when that knowledge is not necessarily shared. Proofs also typically require a brief preamble (a statement of what is to be shown), laying out all premises explicitly, and a statement at the end closing the proof. Typically, proofs are also done in the most general form possible, unless the statement of the proof to be shown includes specifics. This last condition is another common error students make, when asked to show something general, they choose only a specific example to prove. You can't use specifics unless they are provided in the statement of the problem.

To illustrate what is expected stylistically, I'll begin with a couple of examples that don't involve terribly advanced mathematics. The first example is a simple algebra proof for the quadratic equation, and the second will be an example involving the definition of a limit.

Example 1. Prove that the solution to a quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers, has a solution of the form $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

There are two possible methods of doing this proof as stated: 1) one can derive the formula from first principles (i.e. basic algebra steps), or 2) one can replace the form of the solution into the given equation and show that it produces a true statement. This method also requires basic algebra steps, at least in principle, but squaring the solution for x isn't going to be terribly easy, and it's only possible because the solution is proposed in the statement of the proof.

Remember that the details of the individual proof here are less important for illustration purposes than the style and the kind of information you need to include when doing your own.

Method 1.
Consider the quadratic equation $ax^2 + bx + c = 0$, where the coefficients a, b, and c are real numbers. We can find a solution to the equation by completing the square.

We begin by dividing the equation by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Then we move the constant to the right side of the equation: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Recall that a perfect square trinomial with a leading coefficient of one is of the form $(x + k)^2 = x^2 + 2kx + k^2$. Thus to find the value of k, we let $2k = \frac{b}{a}$, or $k = \frac{b}{2a}$. To complete the square then we add $k^2 = \frac{b^2}{4a^2}$ to both sides of the equation: $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$. We then rewrite the left side of the equation as $(x + k)^2$ or $\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$.

Before solving for x, we find a common denominator on the right side. We this by multiplying $-\frac{c}{a}$ by $\frac{4a}{4a}$.

$$\left(x+\frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Taking the square root of both sides we obtain: $x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{2a}$. Finally, subtract $\frac{b}{2a}$ from both sides and write as a single fraction.

$$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

This is the solution that was to be shown. ■

Note: at the end of proofs, typically a symbol like ■, //, or Q.E.D. (which stands for the Latin phrase *quod erat demonstrandum* "thus it has been shown") to indicate that the proof is complete, and that what comes after is not part of the proof. Also, while the algebra is pretty trivial here for someone doing advanced mathematics, because the proof itself is proving something relatively basic, it is expected that the detail is necessary for someone reading the proof. For someone who doesn't need this level of detail, the proof would also not be necessary.

Method 2.

To prove that the solution $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ satisfies the equation $ax^2+bx+c=0$, we need only substitute the proposed solution into the equation and show that a true statement results.

To begin, let's consider the value of $x^2=\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)=\frac{b^2\mp2b\sqrt{b^2-4ac}+b^2-4ac}{4a^2}=\frac{2b^2\mp2b\sqrt{b^2-4ac}-4ac}{4a^2}$ which we obtain by using the formula $(x+k)^2=x^2+2kx+k^2$.

Insert this expression, along with the expression for x into the original quadratic equation:

$$a\left(\frac{2b^2\mp2b\sqrt{b^2-4ac}-4ac}{4a^2}\right)+b\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)+c$$

Here we distribute and separate into individual fractions:

$$\frac{b^2}{2a}\mp\frac{b\sqrt{b^2-4ac}}{2a}-c-\frac{b^2}{2a}\pm\frac{b\sqrt{b^2-4ac}}{2a}+c=0$$

All like terms do indeed cancel, thus the proposed solution does satisfy the quadratic equation. //

Example 2. Prove that the $\lim_{x\to4}f(x)=13$ where $f(x)=3x+1$ using the definition of the limit.

Here we are given a specific function to prove something about. For the unfamiliar, the definition of a limit is that $\lim_{x\to c}f(x)=L$ is that if $|x-c|<\delta$ then $|f(x)-L|<\varepsilon$.

This proof like some proofs must be developed in the reverse order we will have to establish the proof in. We can use algebra to find the value of the limit and use that to give us the relationship we need between ε and δ. Then, we will restart with the "if" premise and show that the "then" conclusion follows from it.

Work needed for the proof, but not to be included in it.

$$\lim_{x \to 4} 3x + 1 = 3(4) + 1 = 13$$

13 is thus the value of the proposed limit. We replace this and the specific function into the expression $|f(x) - L| < \varepsilon$.

$$|3x + 1 - 13| = |3x - 12| = 3|x - 4| < \varepsilon$$

Divide both sides of the last inequality by 3.

$$|x - 4| < \frac{\varepsilon}{3}$$

But since $|x - 4| < \delta$, allow $\delta = \frac{\varepsilon}{3}$. This is the relationship we need for the proof.

Proof.
Let us suppose that $|x - 4| < \delta$. We wish to prove that $|(3x + 1) - 13| < \varepsilon$. To show this, suppose that $\delta = \frac{\varepsilon}{3}$. Then $|x - 4| < \delta$ implies $|x - 4| < \frac{\varepsilon}{3}$. Multiplying both sides of the expression by 3, we obtain $3|x - 4| < \varepsilon \rightarrow |3x - 12| = |3x + 1 - 13| < \varepsilon$ which is what was to be shown. Thus, by the definition of the limit, $\lim_{x \to 4} 3x + 1 = 13$. //

As you can see, proofs need not be long.

Example 3. Prove that $\frac{d}{dx}[\tanh(x)] = sech^2(x)$.

This is another example that we will do in two different ways. Which method is preferred depends on what assumptions you are allowed to make. A shorter version will depend on knowing the derivatives of sinh(x) and cosh(x), presuming that this is known to the reader of the proof or has been previously proved. The longer version will require mostly the same calculus, but will use the exponential definition of tanh(x) and thus will require more algebra. Both methods will ultimately get you to the same result. I will present the longer version first since it requires fewer assumptions.

Method 1.
The definition of $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, and $\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$. We will use these definitions in the proof, together with the general knowledge of differentiating exponential functions.

Thus, $\frac{d}{dx}[\tanh(x)] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]$. To calculate this, we employ the quotient rule. $\frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] = \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - \frac{d}{dx}(e^x + e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = sech^2(x)$. Q.E. D.

You'll notice that this proof used few words, but proofs always employ some words. *Proofs are never bare calculations.*

Method 2.
The last method employed a lot of algebra to use the exponential definition of the hyperbolic tangent. This version will employ knowledge of the fact that $\frac{d}{dx}[\sinh(x)] = \cosh(x)\,, and\ \frac{d}{dx}[\cosh(x)] = \sinh(x)$, together with $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, and $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$.

Since $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$, it follows that $\frac{d}{dx}[\tanh(x)] = \frac{d}{dx}\left[\frac{\sinh(x)}{\cosh(x)}\right]$. Employing the quotient rule, we obtain:

$$\frac{d}{dx}\left[\frac{\sinh(x)}{\cosh(x)}\right] = \frac{\cosh(x)\frac{d}{dx}[\sinh(x)] - \sinh(x)\frac{d}{dx}[\cosh(x)]}{cosh^2(x)} = \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{cosh^2(x)} = \frac{cosh^2(x) - sinh^2(x)}{cosh^2(x)}$$

From this point we must employ an identity for hyperbolic trigonometric functions: $cosh^2(x) - sinh^2(x) = 1$. [This can be easily proved, but should be shown separately.]

Thus, we obtain $\frac{cosh^2(x) - sinh^2(x)}{cosh^2(x)} = \frac{1}{cosh^2(x)} = sech^2(x)$. Q.E.D.

Things to keep in mind when writing proofs.

1. *If you've used no words, your "proof" is not a proof*.
2. *Plan out your proof*. Sometimes you will need to work out certain details in advance first (for instance, as in Example 2, the relationship between epsilon and delta, or in Example 3 Method 2, the identity employed).
3. *Know your audience*. In general, you may assume your audience is the same level as the rest of your classmates. If you were encountering the proof for the first time, what explanations or steps are needed to make the proof clear and easy to read? The most steps you skip, the likely you are to need verbal explanation to support a concise proof.
4. *State your assumptions up front*. Always begin with definitions or any "if"-statements. Proofs frequently begin with "from the definition" or "suppose that…".
5. *Proofs, like well-formed paragraphs have three parts: introduction, body, conclusion*. The introduction includes the statement of the proof to be made, and initial assumptions and definitions. The body carries out the required computations with explanation. The conclusion states the conclusion of the proof, maybe restates what was proved (if the proof is especially long) and marks the end of the proof with a symbol of Q.E.D.
6. *Employ other theorems to make your work shorter*. You can use other theorems in your proofs, as we did with the identity in Example 3, Method 2. If the theorem has a common name, use it. Try not to refer to theorems by the number they appear as in your textbook (since these will differ from book to book or term to term). Instead, briefly state the conclusion of the theorem, for instance, "Since we know by theorem that …", or as we did in Example 3, "by the quotient rule…", which was itself a theorem.