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Math of Finance

I. Simple Interest

$$I = \Pr t$$
$$A = P + I = P + \Pr t = P(1 + rt)$$

Simple interest is calculated just once in a period of time. All the more complicated formulas we use in math of finance are ultimately derived from these simpler cases. When a credit card or car loan company charges you interest each month to update the balance sheet, these are the formulas they use. All of our more complicated formulas will produce identical results if you calculate the formulas for only one compounding period. In general, we wish only to use these formulas when the problem specifies "simple interest".

In our formulas, I is the interest. P is the principle or the beginning balance. A is the amount after interest is added (or the future value). Little r is the annual interest rate. And t is time in years. We can also use r to be the rate per period, and t to be the number of periods, but years is more typical. In solving these problems it can be extremely helpful to make a list of the values for each of the variables in the equation.

Example.

1. Calculate the (simple) interest charged on a credit card loan during a month when the average daily balance is $453.18, and the annual interest rate is 6% annually. Then find the new balance.

   Here, P=$453.18, r=0.06 and t=1/12 since one month is only 1/12 of the year. Thus we get I=453.18*0.06*1/12 = $2.27 rounded to the nearest penny. So the new balance is 453.18+2.27 = $455.45.

This is the simplest case, but, of course, it may be necessary to solve for any of the variables in the equation. A good strategy is to replace values for all the variables that are given, and then solve for the last remaining variable after simplification. This will reduce the need for a set of formulas solved for each variable.

II. Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P(1+i)^m$$

In compound interest, we are calculating interest over several interest periods at once rather than one period at a time, say over a year and once per month, or over 10 years and once per day for all that time. We definitely don't want to do this latter case by hand. Even in Excel, we'd need 3,650 lines of simple interest. Thus, the compound interest formula was derived.

This formula appears in two general forms. In both forms, A is the amount after time has passed or the future value. P is the principle or the present value. No money is added or subtracted in this formula except the interest. In the first form, r is the annual interest rate, and n is the number of times the rate is compounded each year (in the denominator and in the exponent). The t in the exponent is the number of years. In the second formula, r/n is reduced to i, which is the rate per period. And nt is reduced to m, which is the number of times the interest is compounded in total. Problems will usually give n and t separately, but occasionally will state just the total periods.

Examples.

2. Suppose that I want to save money for a down payment on a house in two years. I set aside $10,000 of bonus money in a CD earning 11% annual interest compounded weekly. How much money will I have in my savings for the down payment when the CD matures?

   Here P=10,000, r=0.11, t=2, and n=52. Alternatively, i=0.11/52 and m=104. The formula gives: $A=10000\left(1+\frac{0.11}{52}\right)^{104}=12,457.87$. Thus I'll have $12,457.87 for the payment.

3. Suppose I need $10,000 for a down payment on a new house in five years, but I have only $8,500 to save right now. What annual interest rate will I need to get to save that much in the allotted time?

   Here, we will use the same formula, but we have A=10,000 (notice the future tense), P=8,500, t=5, and r is what we are solving for. Since it's compounded annually, the two versions of the formula are identical. Thus we get: $10000=8500(1+r)^5$. Divide by 8500 on both sides. $\frac{20}{17}=(1+r)^5$. To get inside the parentheses we need to take the fifth root of both sides. $\sqrt[5]{\frac{20}{17}}=1+r\Rightarrow r=\sqrt[5]{\frac{20}{17}}-1\approx 0.033...$ or about 3.3%.

4. Suppose that I have $5000 and I am opening a CD to save money to buy a new car. I can only afford the car I want if I can come up with $7000. The CD pays 8% interest compounded daily. How long will I have to save my money before I have enough to get my car?

   This problem will require a third solution technique, but we have A=7000, P=5000, r=0.08 and n=365. The value of t is what is missing. $7000=5000\left(1+\frac{0.08}{365}\right)^{365t}$. Divide both sides by 5000 to get $\frac{7}{5}=\left(1+\frac{0.08}{365}\right)^{365t}$. Now, in order to get the variable out of the exponent, we will use log properties by taking the natural log of both sides: $\ln\left(\frac{7}{5}\right)=\ln\left(1+\frac{0.08}{365}\right)^{365t}=365t\ln\left(1+\frac{0.08}{365}\right)$. We can bring the exponent down. Now the coefficient of t is 365 and the log term on the right (it's just a constant). Divide both sides by these to isolate t. $\dfrac{\ln\left(\dfrac{7}{5}\right)}{365\ln\left(1+\dfrac{0.08}{365}\right)} = t \approx 4.206...$. Thus, it will take a little longer than four years to get the money we need. Or in days, 1536 days.

Recall that we need to *ROUND UP* to the next complete period regardless of the decimal value. If a hypothetical problem produced 15.01, this means that 15 cycles isn't long enough to meet the required threshold, and so 16 is necessary to achieve it. If the cycles are daily, then this applies to the days, but if the cycle is monthly, it applies to the months.

Also, to avoid rounding errors, carry as many decimals as possible. We want accurate numbers down to the pennies. When we are dealing with figures on the order of $10,000, pennies need at least 7-8 digits to be accurate. When possible, use all the digits in your calculator with the ANS key.

Compounding Periods (n):

a. Annual = 1
b. Semiannually = 2
c. Quarterly = 4
d. Monthly = 12
e. Bimonthly = 24
f. Biweekly = 26
g. Weekly = 52
h. Daily = 365

Other periods are possible, but far less common.

III. Effective Rate

$$r_e = \left(1+\frac{r}{n}\right)^n - 1$$

The effective rate formula is a means by which we can compare rates with different values and different compounding periods. Higher rates are better, but so is more frequent compounding. But what if we are comparing two rates where the higher rate is compounded less often, but the smaller one is compounded more often? The effective rate gives us a value of interest we would earn in a single year, so that we can compare these rates more easily. The effective rate formula is essentially the compound interest formula based on an investment of a dollar, and over the time of one year.

Example.

5. Suppose that we are looking to invest some money in one of two banks. At Bank A they are offering 7% interest compounded daily. At Bank B they are offering 7.25% compounded annually. At which bank should I invest my money?

   I can tell which investment is better by applying the effective rate formula to both banks.

Bank A: r=0.07, n=365. Thus $r_e = \left(1+\frac{0.07}{365}\right)^{365} - 1 \approx 0.07250098...$

Bank B: r=0.0725, n=1. Thus $r_e = \left(1+\frac{0.725}{1}\right)^{1} - 1 = 0.0725$.

It seems that the rates for Banks A & B are virtually identical. They differ only in the 7th decimal place, but Bank A is still slightly better. Depending on how much money one invests and how much time one is saving the money, that small difference may matter.

IV. Compound Interest with Payments

A. Annuities and Loans

$$P = R\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\left(\frac{r}{n}\right)}\right] = R\left[\frac{1-(1+i)^{-m}}{i}\right]$$

As with compound interest, our formula takes two forms, in the one case where r/n is replaced by i (the interest rate over the period) and nt by m (the number of total periods. One major difference here is that R is representing payments rather than principle, and P stands for present value. This formula is used for situations where payments are being applied to a balance owed and the total amount in the account is being paid down. This is the situation where a car or home loan is being paid on, or where an annuity is paying out and the balance is declining (as with lottery winnings or a life insurance policy). Annuities also appear in the savings situation below (B), so be sure you know whether the account is increasing or declining before choosing a formula. The key feature differentiating this formula from the one for savings is the negative exponent indicates that the total is decreasing rather than increasing.

Example.

6. Suppose that I wish to purchase a car that is sticker priced at $15,000. I can get a loan for 5% interest with the credit union. How much money will I have to pay each month if I take out a loan for five years?

   Here P=15,000, r=0.05, n=12 (monthly) and t=5. R is to be solved for. Thus

   $$15000 = R\left[\frac{1-\left(1+\frac{0.05}{12}\right)^{-60}}{\left(\frac{0.05}{12}\right)}\right] \text{ or } \frac{15000}{\left[\frac{1-\left(1+\frac{0.05}{12}\right)^{-60}}{\left(\frac{0.05}{12}\right)}\right]} = R \approx 283.07$$

   When entering this expression into the calculator, it may help to work from the inside of the parentheses and simplify as you go (carrying as many decimal places as possible). This will avoid misplacing some of the parentheses.

B. Sinking Funds and Annuities

$$S = R\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\left(\frac{r}{n}\right)}\right] = R\left[\frac{(1+i)^m-1}{i}\right]$$

When making payments on an account and the value in the account is increasing, this is a sinking fund when it's a savings account, and it may also be the payment processing into an annuity (so that the balance can be built up so that it will eventually pay out as in the previous case). Note the positive exponent that indicates that the balance in the account is growing. The values of the variables are the same as with the previous case except that S stands for the future value.

Example.

7. Suppose a company is saving up to purchase new computers for their office. They've determined that they are going to need $12,000 for the purchase of computer for their whole office three years from now. Their bank is willing to give them an annual interest rate (compounded monthly) of 4.5%. How large a monthly payment must the company make to a sinking fund so that they have enough money for the purchase?

   Our variables are S=12,000, r=0.045, n=12, t=3. Thus we get:

   $$12000 = R\left[\frac{\left(1+\frac{0.045}{12}\right)^{36}-1}{\left(\frac{0.045}{12}\right)}\right]$$, giving $$\frac{12000}{\left[\frac{\left(1+\frac{0.045}{12}\right)^{36}-1}{\left(\frac{0.045}{12}\right)}\right]} = R \approx 311.96$$.

V. Continuous Interest and Effective Rate

A. Continuous Interest

$$A = Pe^{rt}$$

As interest is compounded more frequently, the amount of interest earned (the effective rate) increases. However, it does not increase in an unlimited fashion. If I compound daily I will earn a bit more than if I compound monthly, but if I compound every hour or every second, I will not increase my interest earned by that much. There is a limit to how much interest I can earn even if I compound every nanosecond. That limit is the case of continuously compounding interest. That math to prove this formula is beyond this course, but you can assure yourself of it numerically by taking the compound interest formula from Section II and increase the value of n as much as you want. You will see that that value of the equation approaches the value of the continuous compounding equation above.

Variables in the equation are familiar ones: A is the future value, P the principle, r the interest rate as a decimal and t is the number of years.

Continuous compounding problems generally state that this is the formula to be used. Examples of situations in which continuous compounding is a reasonable model would be the stock market average (long term) or inflation.

Example.

8. Suppose that you are investing 25,000 in an IRA account that is offering continuous compounding at 10% for 15 years. How much will be in the account?

   Using $A = Pe^{rt}$ we get: $A = 25000e^{0.10*15} \approx 112,042.23$

   As with discrete compounding, we can solve for any of the variable in the equation including the rate and the time. In both of these cases, we will use log properties as we did in Example #4.

B. Effective Rate

$$r_e = e^r - 1$$

Just as with discrete compounding, we may wish to compare rates. Perhaps Bank A is offering continuous compounding but Bank B is offering discrete compounding at a higher rate? This is the formula that will allow us to make the right comparison by looking at the interest earned at the end of one year.

Example.

9. Suppose Bank A offers a bank account with continuous compounding at 6% interest. And Bank B across town offers a rate at 6.20% compounding monthly. Which bank you should invest your money in?

   Bank A: $r_e = e^{0.06} - 1 \approx 0.061836...$

   Bank B: $r_e = \left(1 + \frac{0.062}{12}\right)^{12} - 1 \approx 0.063792...$

   According to our calculations, Bank B is better.

Problems.

i. What is the effective rate on the following interest rate scenarios?
   a. 3% compounded quarterly
   b. 5% compounded daily
   c. 6% compounded semiannually
   d. 8% compounded weekly

ii. How much money will be in an account that invested \$6000 for 8 years at a rate of 5% compounded monthly?

iii. Over a 5 year period, a principle of \$2000 accumulated to \$2950. If interest was compounding quarterly, what was the annual interest rate on the account?

iv. Suppose new parents are saving for their child's education 18 years down the road. How much money do they need to set aside now at 7% compounded annually to have \$50,000 for college tuition?

v. If the rate of inflation for certain goods is 7.25% compounded daily, how many years will it take for the average price of the good to double? (Let the current price be P, and the future price be 2P.)

vi. Consider the scenario in problem iv. What if the couple wants to make annual payments on the account instead of one lump sum payment? How large do the payments have to be each year to achieve the $50,000 goal?

vii. A debt of $7000 due in 5 years is to be repaid by a payment of $3000 now and a second payment at the end of five years. How much is the second payment if the interest rate is 8% compounded monthly?

viii. What annual rate compounded continuously is equivalent to an effective rate of 5%?

ix. If interest is compounded continuously at 7% interest, how many years would it take for the investment to triple?

x. What is the present value of an annuity if you are paying $600 for six years at 6% interest compounded annually?

xi. Mary won a state lottery for $4,000,000 and elects to receive a single payment now rather than have payments broken up over 20 years. If the state can invest the winnings at 10% compounded monthly, what is the value of the account Mary should receive now?

xii. In ten years, replacing equipment for a factory will cost $75,000. The old equipment will have a salvage value of $12,000. How much does the company need to set aside in a sinking fund each month if their account earns 4% compounded monthly?

Math of Finance Formula Sheet

$$I = \Pr t$$

$$A = P + \Pr t$$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$r_e = \left(1+\frac{r}{n}\right)^{n} - 1$$

$$A = R\left[\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\left(\frac{r}{n}\right)}\right]$$

$$S = R\left[\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\left(\frac{r}{n}\right)}\right]$$

$$A = Pe^{rt}$$

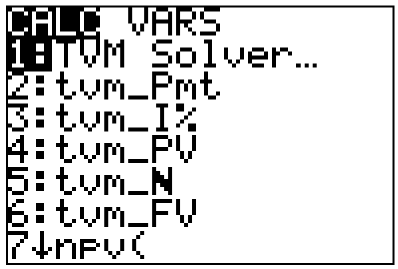
$$r_e = e^{r} - 1$$

Calculator Tips & Tricks

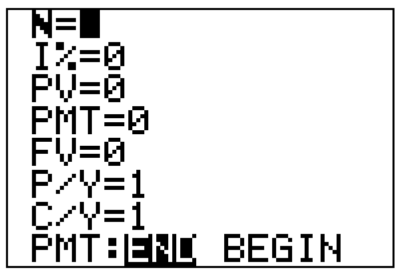
Note: On all exams and quizzes it will be necessary for students to show all work. The calculator techniques described here are not meant to replace by-hand calculations, but only to check your work. Students who provide only an answer on exams or other assessments will receive no credit.

In your TI-83/84 calculator, most of them have a FINANCE function. Mine is above the x^{-1} key, but depending on how old your calculator is, it may be under the Program menu. Once into the program, it works the same in all calculators.

The screen you get should look like this:



Choose the TVM Solver…



On this screen you will enter all the values for the equation.
N is the total number of compounding periods for the calculation (in our formulas this was nt or m).
I% is the interest rate in percentage form; do not convert it to a decimal as the program will do that for you.
PV is the Present Value, typically the principle or P in our equations.
PMT is the payment line, or R in our equations.
FV is the future value; A or S in our equations.
P/Y is the payments per year.
C/Y is the compounding per year; n in our equations. Note that payments per year and compounding per year should be the same value. If you change P/Y, C/Y will change automatically.
PMT: END BEGIN determines whether interest is earned at the end of the period or at the beginning. Only use BEGIN for Annuity Due. All others use END. (Don't forget to change it back!)

Notes on use.

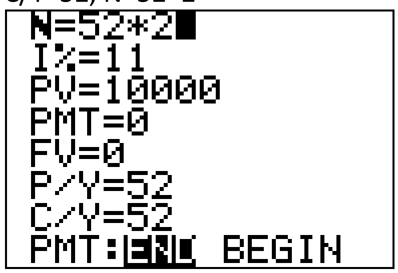
a. Only ever use two of PV, PMT or FV. You should never have all three with values. One must be zero.
b. You can do calculations on screen such as monthly payments for 7 years you can enter on the N line as 12*7 and the calculator will calculator the value for you.

c. The program has a quirk in that when it does the calculation PV, PMT or FV will have one of the values you use typically come up negative. You can ignore this in these locations generally speaking: it is a quirk of the program. However, you will have to enter the negative on one of them if you wish to solve for N or I%. The answer won't come out right if you don't.

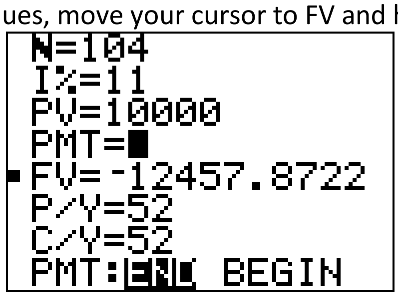
Examples.

10. (Example 2) Suppose that I want to save money for a down payment on a house in two years. I set aside $10,000 of bonus money in a CD earning 11% annual interest compounded weekly. How much money will I have in my savings for the down payment when the CD matures?

    PV=10,000, I%=11, P/Y=C/Y=52, N=52*2

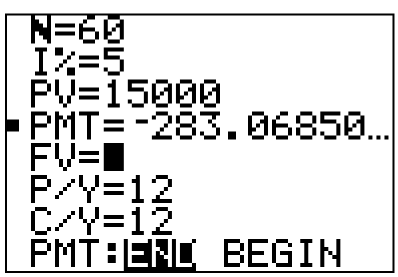
    After entering these values, move your cursor to FV and hit ALPHA ENTER for "Solve".

    Notice the negative sign in front of the future value? It's an artifact of the program. The rest of the number is identical to what we got when we solved Example 2 by hand.

11. (Example 6) Suppose that I wish to purchase a car that is sticker priced at $15,000. I can get a loan for 5% interest with the credit union. How much money will I have to pay each month if I take out a loan for five years?

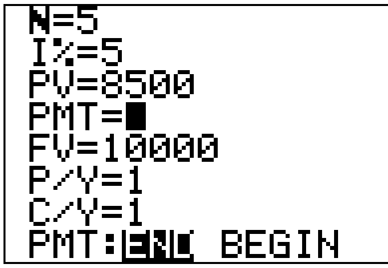
    Here, PV=15,000, I%=5, P/Y=C/Y=12, N=5*12 (Reset FV to 0.)
    As before, move your cursor to the line you are solving for (here the PMT), and then hit ALPHA ENTER.

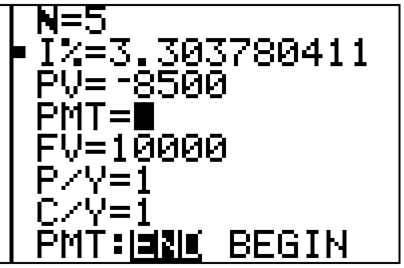
    

    Again notice the negative sign? But the value is the same that we found doing the problem by hand, rounded to the nearest penny.

12. (Example 3) Suppose I need $10,000 for a down payment on a new house in five years, but I have only $8,500 to save right now. What annual interest rate will I need to get to save that much in the allotted time?

    Here we wish to solve for the interest rate. Now, we will have to enter that negative sign to get the correct answer. Enter the following values: FV=10,000, PV=8500, N=1*5, P/Y=C/Y=1. Reset the PMT to zero.

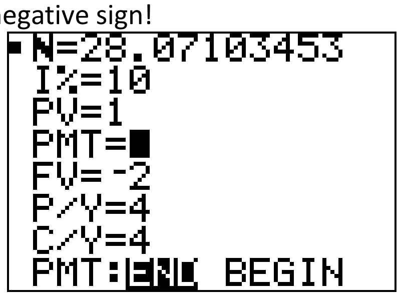
    

    If you enter the information as I did here, you will actually get an error when you solve for I%. However, if we insert the negative sign on either PV or FV we get the second screen. This is the same value we found when we did this problem by hand. (It does not matter whether you enter the negative one the PV or the FV, just that they have to be different signs.) You may not get an error depending on the values, but a negative interest rate is not good, and will be incorrect.

13. Let's suppose I want to know how long it will take an investment to double if I am earning 10% interest compounded quarterly.

    We know that P/Y=C/Y=4, and that I%=10, but we don't know PV or FV. But this is a doubling problem, so to use the calculator, we can choose any initial investment we want, even a dollar. The time to doubling will be the same regardless. Then we will solve for N. Remember to use the negative sign!

    Without the negative sign, I got another error, but that won't always happen. Remember that negative N's are backwards in time, and it's a sign there is a problem. Here, we got 28.07 **quarters**. Remember to round up to 29 quarters, and then you can convert to years, which would be 7.25 years.

Remember, you can use this to check your work, but you must show all hand calculations on tests and quizzes. Getting the wrong information out of the calculator because you misused it is no reason for me to give back lost points.