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Representing 3D surfaces in 2D graphs

In order to work with three-dimensional surfaces, we need to be able to visualize those graphs in some form that will allow us to work with them. Most people aren't great artists, so we will be concentrating here on representing three-dimensional graphs as two-dimensional graphs. Each technique will be useful in different contexts, but all will provide information useful for us to learn to visualize the shape of the surface, and eventually, set up limits for double and triple integrals, and finding intersections of such graphs.

Level Curves.

The first technique we are going to consider is drawing level curves, also called contour curves. This is the best technique for representing all three dimensions in a plane, and is commonly used in map-making. The technique involves essentially taking cross-sections of a three-dimensional curve at selected constant *z*-values, and plotting the resulting two dimensional function onto the plane. Knowing in which direction among that graphs that *z* is increasing, allows us to mentally translate these planar cross-sections into some notion of the general shape of the surface. The graphs we obtain are what we would see if we marked these constant z-values on the surface, and then looked down on the surface from above, looking down the z-axis.

Let us consider an example so that we can go through the technique for creating these graphs, and then compare the level curve graph to the three-dimensional graphs. You will find that you will be better able to make the mental translation by using a three-dimensional grapher to compare your results obtained through level curves to the true graph for a number of examples.

Example 1. Sketch at least 6 level curves for the graph $g(x,y)=\frac{8}{1+x^2+y^2}$.

We begin by noting the range of the function. It has a maximum of 8 (when (x,y)=(0,0)) and a minimum approaching but never reaching zero: $z\in(0,8]$. We should choose our six *z*-values in this range. Let's choose: 8, 4, 2, 1, ½, ¼, 1/8. (I've chosen 7 values here since as we will see, z=8 will produce only a dot rather than a curve.)

To reduce the amount of algebra we will have to do, let us do the algebra before replacing **z** in the equation. Solve the equation $c=\frac{8}{1+x^2+y^2}$ for y, where *c* stands in for our *z*-values, and reminds us that we are holding *z* constant for each curve.

$$c=\frac{8}{1+x^2+y^2}\to 1+x^2+y^2=\frac{8}{c}\to x^2+y^2=\left(\frac{8}{c}-1\right)$$

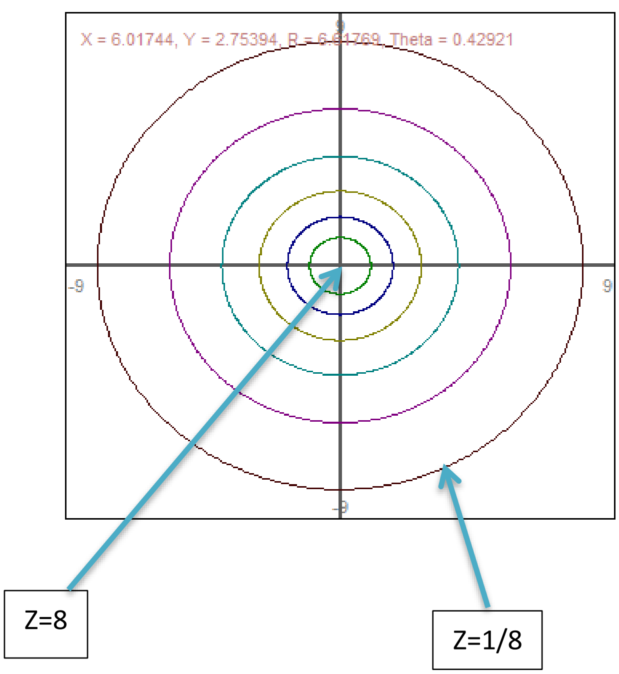
We can stop before we get all the way to *y=f(x)* if we, as we did here, obtain a common graph in standard form. This is a circle of radius $\sqrt{\frac{8}{c}-1}$, and further solving for $y=\pm\sqrt{\frac{8}{c}-1-x^2}$ will double the number of graphs we will need to draw to get all the level curves since each curve is represented by two functions, or else you can convert to polar for such graphing problems. We will need to go all the way if we are using a calculator to obtain the graphs, however. For now, we will stick with the standard equation of the circle: $x^2 + y^2 = \left(\frac{8}{c} - 1\right)$.

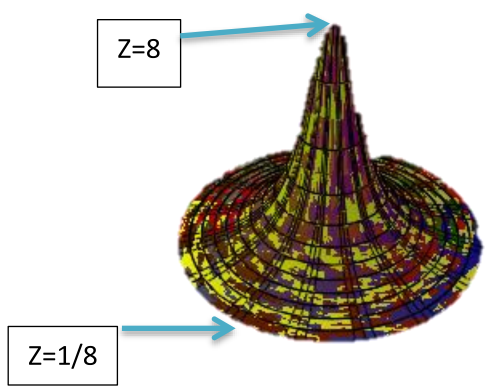
Next, replace the chosen values of c (our z-values) to obtain the set of two-dimensional equations to be graphed.

$$c = 8: x^2 + y^2 = \left(\frac{8}{8} - 1\right) = 0$$
$$c = 4: x^2 + y^2 = \left(\frac{8}{4} - 1\right) = 1$$
$$c = 2: x^2 + y^2 = \left(\frac{8}{2} - 1\right) = 3$$
$$c = 1: x^2 + y^2 = \left(\frac{8}{1} - 1\right) = 7$$
$$c = \frac{1}{2}: x^2 + y^2 = \left(\frac{8}{\frac{1}{2}} - 1\right) = 15$$
$$c = \frac{1}{4}: x^2 + y^2 = \left(\frac{8}{\frac{1}{4}} - 1\right) = 31$$
$$c = \frac{1}{8}: x^2 + y^2 = \left(\frac{8}{\frac{1}{8}} - 1\right) = 63$$

Thus we obtain a series of concentric circles with radii equal to $\{0, 1, \sqrt{3} \approx 1.7, \sqrt{7} \approx 2.6, \sqrt{15} \approx 3.9, \sqrt{31} \approx 5.6, \sqrt{63} \approx 7.9\}$

Place all these circles on a single graph (if possible).





Graphing circles in polar coordinates may be of use here. The circle of radius zero is just a dot at the origin, since that is the only pair of *(x,y)* that satisfies the equation $x^2 + y^2 = 0$.

What are we to read from this graph? Looking down the *z*-axis, we see a series of concentric circles, circles that rapidly spread further and further apart as we approach the *xy*-plane. Compare this to the three-dimensional image of the graph.

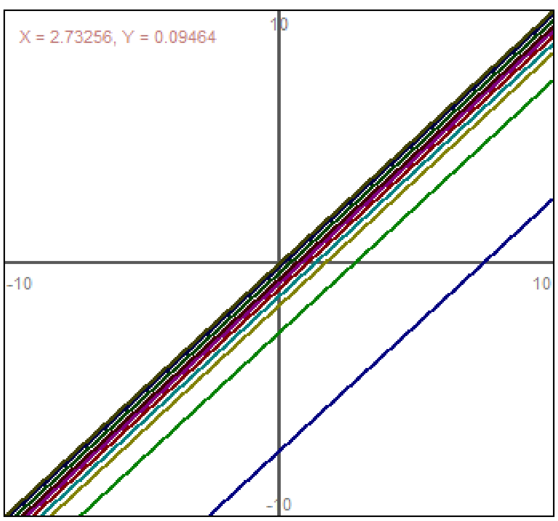
Let's consider another example that doesn't involve concentric circles.

Example 2. Draw ten level curves for the graph $f(x,y) = \ln(x-y)$.

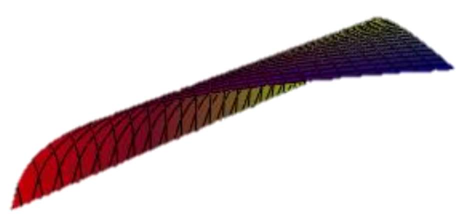
Again, consider the range of the function to choose the values for *z* to draw. The natural log has a range from $(-\infty, \infty)$, so we should choose both positive and negative values, and for the sake of simplicity, around the origin: $\left\{0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2,\right\}$. Replacing *f(x,y)* in the equation for *c*, solve for *y*. This list only gives us nine level curves, so since natural log has an asymptote, plot that one as well.

$$c = \ln(x-y) \rightarrow e^c = x - y \rightarrow y = x - e^c$$

Here we have a linear equation representing the level curves, and the *z*-value is changing the value of those intercepts. Collecting all these lines in one graph we get:

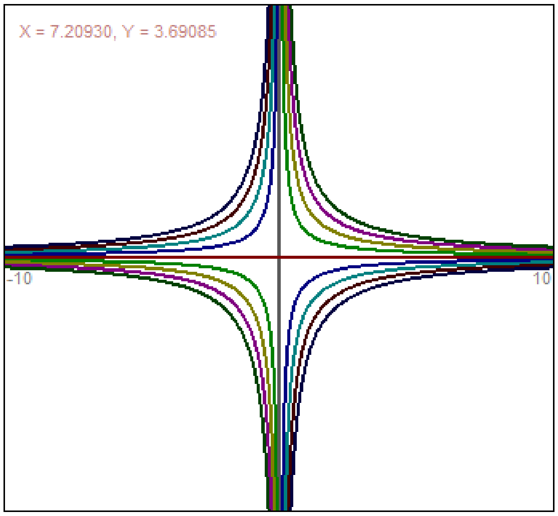


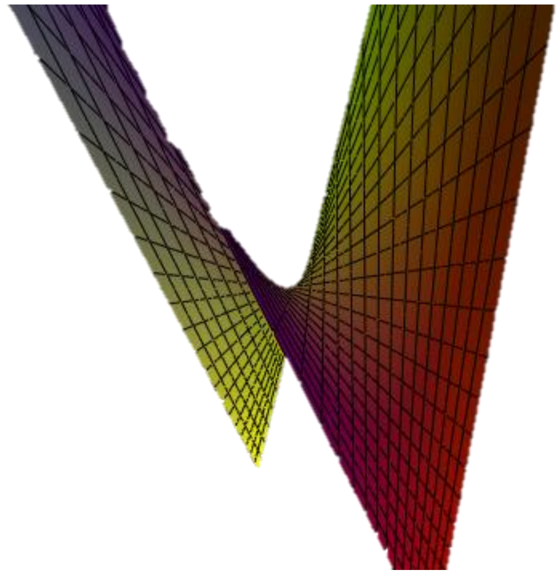
Notice how the lines here are all getting progressively closer together until they reach the asymptote at $y = x - e^{-\infty}$ or *y*=*x*. This graph reminds me of a waterfall with the *z*-values plunging very rapidly over a cliff as you approach the asymptotic line. Consider what this looks like in three-dimensions and compare.



Practice Problems.

1. Explain in your own words how the graph *z*=*xy*, shown here, is represented by the level curves shown.





[Hint: the first and third quadrants are representing positive z-values, while the second and fourth quadrants are representing negative *z*-values.]

2. Draw 6-10 level curves for each of the following functions. Use a 3D graphing program to compare your results to the original graph. State the range of *z* for each problem and choose *c*-values accordingly. Be sure to label the extremes of the graphs to see the direction of z.
   a. $z = 6 - 2x - 3y$
   b. $f(x, y) = \sqrt{9 - x^2 - y^2}$
   c. $g(x, y) = e^{xy/2}$
   d. $h(x, y) = \frac{x}{x^2+y^2}$
   e. $z = 16 - 2x^2 - y^2$
   f. $p(x, y) = 4\sin(\mathrm{x} + 3\mathrm{y})$
   g. $z = 2rsin^2\theta$ [Hint: this works like the others, but solve for *r* instead to graph level curves in polar coordinates.]

3. We can extend the notion of level curves into four-dimensional equations and graph level surfaces in three-dimensions so see how the graph changes as the value of the function changes. This is one of the ways we can mentally represent four-dimensional functions easily. Use the technique outlined here to graph 6 level surfaces of the function $w = \sqrt{x + y + z}$ by choosing appropriate values for w and graphing the resulting six level surfaces in sequence from smallest *w*-value to the largest.

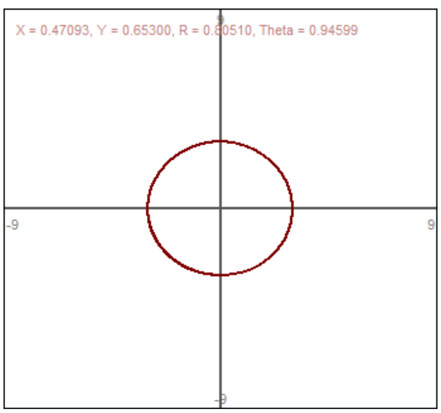
In addition to using level curves, we have some other means of looking at functions that are really in three-dimensions by considering them in two-dimensions. Such a method is that of projection.

Particularly when we are interested in setting up double and triple integrals, we are primarily interested in the orientation of functions, and then how they project at their widest point onto the plane of the remaining variables. It's a bit like looking at a shadow of the graphs. If we shine a light along the axis we are removing, what does the shadow of the graph(s) look like projected against a wall? At other times, we will have three-dimensional graphs defined only in terms of two variables. In these cases, the graph will extend out of the paper along the curve formed by the two-dimensional graph in a sheet (if we are in rectangular coordinates) or as a rotation of the graph drawn (if we are in polar coordinates).

Example 3. What is the projection onto the *xy*-plane of the sphere $x^2 + y^2 + z^2 = 9$.

If we are projecting onto the *xy*-plane, then we need to eliminate the *z*-variable. To do this properly, we need to choose the *z*-value to insert into the equation where the graph is widest. In the case of the sphere, this is the value of *c* where the plane *z=c* passes through the center of the sphere. Since the center of this sphere is (0,0,0), we choose *z=0*.

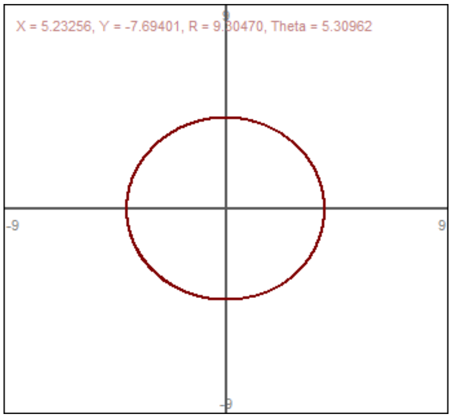
This leaves us with the graph in two-dimensions $x^2 + y^2 = 9$, and so this is the projection we wish to draw to represent this sphere in the plane.

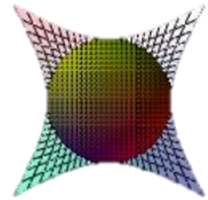


In fact, because of the symmetry of the sphere, this is the same graph we would obtain regardless of which dimension we omitted.

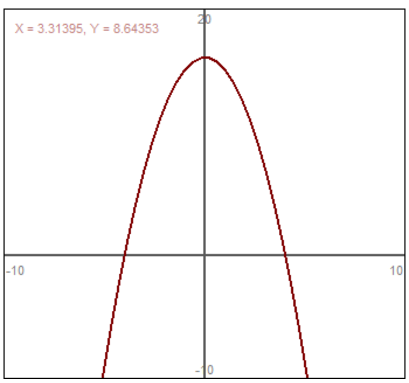
Example 4. Project the graph of the paraboloid $z = 16 - x^2 - y^2$ onto a) the *xy*-plane, for the portion of the graph above *z=0*; and b) onto the *xz*-plane, c) onto the *zr*-plane.

For part *a*, we first consider the widest part of the graph. Since this graph is a paraboloid opening down, the widest part of the graph will be at the lowest *z*-value. Since this is specified to be *z=0*, we use that value. Moving the variables to the other side of the equation, we obtain the projection: $x^2 + y^2 = 16$, which is a circle of radius 4.





For part *b*, if we want to project onto the *xz*-plane, we need to remove the *y*-variable from the equation. As with the sphere, we do this by finding out where the widest point of the graph is in the *y*-direction. We do this by determining where the vertex of the paraboloid is, which is at (0,0,16), so we choose *y=0*. This results in the equations in only two variables: $z = 16 - x^2$. By treating *z* as we would *y* in a normal 2D graph (i.e. the vertical axis is *z*), we can obtain the projection into the *xz*-plane.



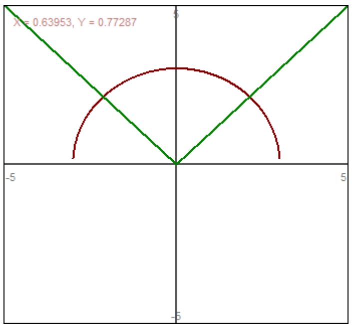
It should be noted that this type of representation loses a lot of information about *y*. This is the same graph we would get if we projected $z = 16 - x^2 - (y-3)^2$ or $z = 16 - x^2 - 4y^2$, or $z = 16 - x^2 + y^2$ onto the *xz*-plane. However, these graphs would look different if we projected onto the *xy*-plane or the *yz*-plane, so it often is necessary to plot graphs in more than one two-dimensional format if we wish to tell them apart from each other. Parts *a* and *b* here would be enough to set up a triple integral for the volume under the paraboloid, as we will see when we get to that section.

Lastly, part *c* can be done in this instance because the graph is symmetric in *x* and *y*, and converts easily to a polar graph in two variables, *z* and *r*. $f(r,\theta) = 16 - r^2$. Since converting to polar leaves us with no dependency on θ, so again, we'd graph this as we would normally in *x*-*y*, but here *r* will be the horizontal axis, and the vertical axis is *z*. Indeed, if you want to graph this in your calculator, simply do the replacement *y* for *z* and *x* for *r* and graph $y = 16 - x^2$, which is the same graph as that which we obtained in part *b*. Here, though, we know that our three-dimensional graph represents a rotation of this graph around the *z*-axis.

This procedure can be done for two functions at once as well, in order to obtain orientation and intersection information.

Example 5. Draw a projection graph of the intersection and orientation of the two surfaces $z = \sqrt{9 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$.

Let us first consider a projection onto the *xz*- or *yz*-plane. Since the graphs are symmetrical in these coordinates, both graphs will look the same, thus I will arbitrarily choose the *xz*-plane. The widest part of both graphs goes through *y=0* since both are centered at the origin, resulting in two equations in *x* and *z* only: $z = \sqrt{9 - x^2}$ and $z = \sqrt{x^2} = |x|$. Be careful about reducing this last equation, because the absolute value is needed here to obtain the proper graph.



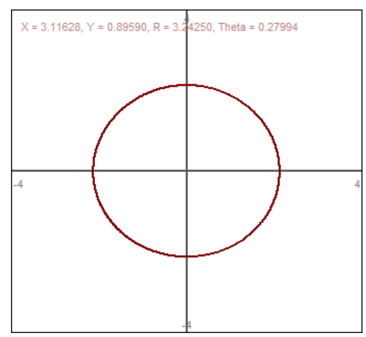
Note that our original graphs where those of a sphere and the top half of a cone.

Now, to obtain the projection of their intersection, we need the widest part of the region. This occurs, as it commonly does (though not universally) at the intersection. We can solve for the equation in two dimensions by equating the original two equations.

$$\sqrt{9 - x^2 - y^2} = \sqrt{x^2 + y^2}$$

$$9 - x^2 - y^2 = x^2 + y^2 \rightarrow 9 = 2x^2 + 2y^2 \rightarrow \frac{9}{2} = x^2 + y^2$$

So, we obtain a circle of intersection of radius $\frac{3}{\sqrt{2}}$.



Practice Problems.

4. Draw two projections of each graph or set of graphs. In each case you can choose from three options: *xy*-plane, *xz*-plane, *yz*-plane for rectangular graphs, and *r*θ-plane, *rz*-plane and θ*z*-plane for polar graphs, and similarly ρφ-plane, ρθ-plane, and θφ-plane for spherical graphs. Treat ρφ- and ρθ-planes like *r*θ-planes, but you should treat θφ-planes as *xy* rectangular planes.
   a. $z = 6 - x^2 - y^2, z = -\sqrt{4 - x^2 - y^2}$
   b. $z = 4 - x^2, z = 0$
   c. $z = x^2 + 3y^2, z = 9$
   d. $\frac{x^2}{4} + \frac{y^2}{16} + z^2 = 1$
   e. $z = e^{-r^2}$
   f. $z = \frac{8}{1+r^2}$
   g. $\rho = sec\varphi$
   h. $\rho = 4sin\varphi$