Limits of Integration in 2D & 3D

When working with integrating in multiple variables, sometimes we will need to change the order of integration provided to us. This is related to the procedure for setting up integrals in a convenient order, particularly if we wish to reduce the number of integrals we'll need to complete in some cases. We will also consider changing variables, from rectangular to polar/cylindrical, or rectangular to spherical.

Let's begin with an easy case of changing the order of integration for two variables.

Example 1. Switch the order of integration for $\int_{-1}^{1}\int_{x^2}^{1} xy\, dydx$.

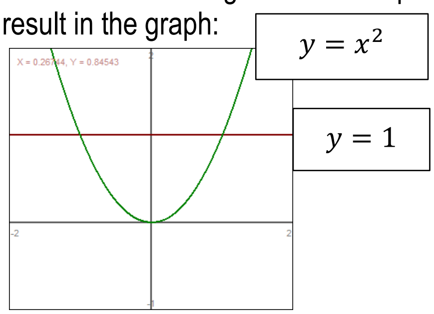
In order to change the order of integration, we need to look at the graph of the region in the xy-plane that the limits describe. We can break down the integral limits into equations:

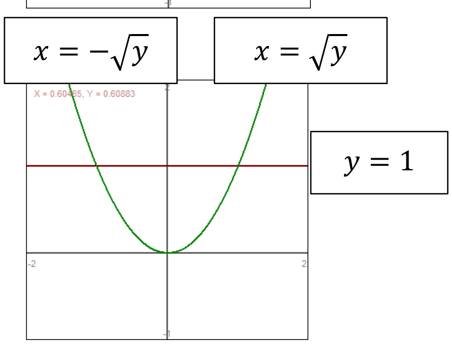
$x = 1$ $\quad$ $y = 1$

$$\int_{-1}^{1}\int_{x^2}^{1} xy\, dydx$$

$x = -1$ $\quad$ $y = x^2$

We take the missing side of the equations from the variable of integration for those limits. These equations result in the graph:





The equations $x = 1$ and $x = -1$ are the intersections of these two graphs. When in doubt, begin graphing from the inside limits, and eliminate any intersections so that they are not considered separate functions.

In order to change the order of integration, we need to change the variable we have solved for. Therefore, consider the graph relabeled.

The inside limits of integration are the right and left functions which go on the inside limits. The top and bottom values go on the outside limits. Here, the minimum value for y is $y = 0$. This results in the integral now:

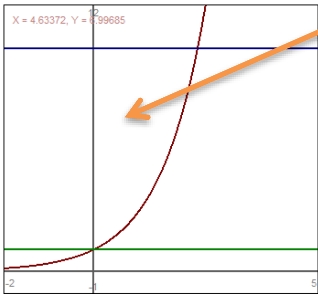
$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} xy\, dxdy$$

Note carefully: we did not change the function inside the integral, nor did it factor into any of our graphs. It is not relevant to graphing the region or related in any way to the limits of integration.

Let's consider a second problem.

Example 2. Switch the order of integration in $\int_1^{10}\int_0^{lny} dxdy$ to simplify the integration.

As before, begin by graphing the equations in the limits. Specifically, we have $x = \ln y$ and *x = 0* from the inside limits, and *y = 1*, and *y = 10* from the outside limits. To graph these, it may be useful to solve the equation $x = \ln y$ for *y* to obtain a more familiar function $y = e^x$.



The region we are integrating over is here. Note that *y = 1* is a point of intersection of *x =0* and $y = e^x$.

The new order of integration will have *y* as the inside limits, and *x* on the outside. In the *y*-direction, the top function is *y = 10*, and the bottom function the curve $y = e^x$. In the x-direction, the limits are the leftmost value of *x = 0*, and the rightmost value obtained from intersecting *y = 10* and $y = e^x$, which is *x = ln(10)*. This gives us:

$$\int_0^{\ln 10}\int_{e^x}^{10} dydx$$

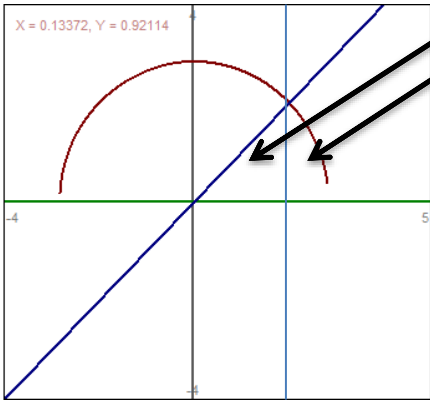
Practice Problems.
Change the order of integration for each of the following problems. Sketch the graph of each region.

1. $\int_0^1\int_x^1 x\sqrt{1+2y^3}dydx$
2. $\int_0^1\int_{3x}^3 6e^{y^2}dydx$
3. $\int_0^9\int_{\sqrt{x}}^3 \frac{4}{5+y^3}dydx$
4. $\int_0^4\int_x^4 e^{-y^2}dydx$
5. $\int_0^{\sqrt{\frac{\pi}{2}\pi}}\int_y^{\sqrt{\frac{\pi}{2}}} \sin x^2\, dxdy$
6. $\int_0^1\int_{y^2}^1 \sqrt{x}\sin x\, dxdy$

Before going on to triple integrals, let's consider changing variables to from rectangular to polar coordinates.

Example 3. Rewrite the integrals $\int_0^2\int_0^x xy\, dydx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} xy\, dydx$ as a single integral in polar coordinates.

We can see that this may be possible, since the function to be integrated in both integrals is the same. We need to begin by graphing the functions. From the integrals we obtain the equations $y = 0, y = x, y = \sqrt{8-x^2}$. This last equation can be rearranged into the equation of a circle $x^2 + y^2 = 8$. By finding the intersections of these equations, we can determine which of the outside limits are needed. $y = x\ and\ y = \sqrt{8-x^2}$ intersects at x = 2. $y = 0\ and\ y = \sqrt{8-x^2}$ intersect at $x = \sqrt{8} = 2\sqrt{2}$. The intersection of *y = 0* with *y = x* provides the intersection *x = 0*. The region is plotted below.



The first integral is shown on the left, while the second integral is the section on the right.

We can see, however, that in polar coordinates, this is a single sector of a circle. The circle is of radius $2\sqrt{2}$. What we will need to find is the angle for θ. It's a portion of the first quadrant. Since *x = 0* is a boundary, θ = 0, is one limit. What we need to find is the other.

Equations of lines that pass through the origin will produce an angle for θ using the equation $\theta = tan^{-1}\left(\frac{y}{x}\right)$. Solving the equation *y = x* for *y/x*, we get $\frac{y}{x} = 1$. Replacing this, we get $\theta = tan^{-1}(1) = \frac{\pi}{4}$.

Since it's typical to integrate in r first, we will follow suit here, and obtain the integral $\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} f(r,\theta)rdrd\theta$. The function we are integrating can itself be found by substituting $x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ as appropriate, to finally obtain $\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} r^3\sin\theta\cos\theta\, drd\theta$.

The procedure followed here will be basically the same for problems that have only one region in rectangular coordinates, save for not needing to combine regions from two integrals. In Example 3, we ended up with limits that were all constant, but this is by no means guaranteed. If you have functions bounding your region that do not have simple and common polar analogues, use the conversion formulas on the boundaries, and recall that typically *r* does not take on negative values (thus it starts at 0 unless there is a function or another positive value to use). Not all graphs require going all the way to 2π in θ to sweep through the entire graph. You will want to review polar graphs at this point for the less common graphs.

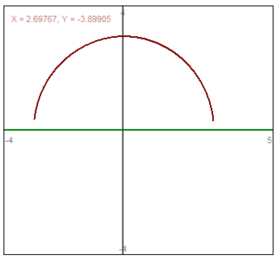
Practice Problems.
Convert the following graphs from rectangular into polar coordinates. Sketch the graph of each region.

1. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} x^2 + y^2 dydx$
2. $\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2)^{3/2} dydx$
3. $\int_0^2\int_y^{\sqrt{8-y^2}} \sin\sqrt{x^2+y^2}\, dxdy$
4. $\int_0^6\int_0^{\sqrt{6y-y^2}} x^2 dxdy$
5. $\int_0^2\int_0^x xydydx$

Dealing with these same issues in three variables is fraught with problems. There are cases where we can reduce the problem to a two-variable case, and so we will start with those.

Example 4. Change the order of integration in the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{6-x-y} dzdydx$ to the order *dzdxdy*.

Since the difference between the two orders of integration, does not change the inside limit, we do not have to change the inside integral at all. That leaves us with just the two variables in the outside two integrals. This case will be just like the two variable case. We are given the equations $y = 0 \; and \; y = \sqrt{9 - x^2}$, which are the bounds on the top half of the circle. They intersect at *x* = *3*, and so the third boundary is *x* = *0*, giving us a region in these two dimensions that is just the first quadrant of a circle.



If we solve the equation of the circle for *x* instead of *y*, we get $x = \sqrt{9 - y^2}$, and *x* = *0* also goes in the new inside limits for *x*, and finally, these intersect at *y* = *3*, and the other boundary *y* = *0*.

Our integral then is $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{6-x-y} dzdxdy.$

Example 5. Change the order of integration in the integral $\int_0^4 \int_0^{4-x/2} \int_0^{12-3x-6y/4} dzdydx$ to the order *dydxdz*.

Converting three variables simultaneously is problematic, since we need to visualize the region under consideration. A three-dimensional graphing program can be helpful here. This region is just the volume under the plane $z = \frac{12-3x-6y}{4}$, with intercepts (4,0,0), (0,2,0), and (0,0,3). This is a well-behaved region with only one function (the plane) bounding the region in the positive direction and the coordinate plane in the other. The equations *x* = *0, y* = *0, z* = *0* are the dead giveaways for coordinate planes in triple integrals. The equation $y = \frac{4-x}{2}$ is just the intersection of *z* = *0* and the plane, so all the information we have comes from just four equations. For a problem like this, solve the plane for the variable you wish to begin with, here *y*: $y = \frac{12-3x-4z}{6}$. This, together with *y* = *0* are the limits for *y*. The intersection of these two lines gives us an equation in *x* and *z* only. $3x + 4z = 12$. Our order of integration requests that *x* be next, so we solve this equation for *x*: $x = \frac{12-4z}{3}$. This and *x* = *0* are limits in the middle integral. And set these equal to each other to find the limit in *z*, but it will be the *z*-intercept mentioned earlier at *z* = *3*. This gives us the final integral $\int_0^3 \int_0^{12-4z/3} \int_0^{12-3x-4z/6} dydxdz.$

Practice Problems.
Change the order of integration of the given integral to the specified order.

1. $\int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} dzdxdy$ in the order dzdydx
2. $\int_0^1 \int_{-1}^0 \int_0^{y^2} dzdydx$ in the order dydzdx
3. $\int_0^2 \int_{2x}^4 \int_0^{\sqrt{y^2-4x^2}} dzdydx$ in the order dxdydz

Converting a problem to cylindrical coordinates, when *dz* is the inside limit, can be reduced merely to the two variable problem. The function and the limits for *z* can just be replaced using polar equivalencies, and then the remaining two limits can be done, similar to what we did in Example 4, as for two variables. Graph the limit functions and convert the region graphically as we did in Example 3. For this reason, the next example I will do will be in spherical coordinates.

Example 6. Convert the integral $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-y^2-x^2}} \sqrt{x^2+y^2+z^2}\,dzdydx$ to spherical coordinates.

If we begin with the inside limits, we find that the upper limit is the top hemisphere of a sphere of radius 3. We always start by converting the inside equations into spherical, and then check which of the remaining equations are intersections or which are independent boundaries. The bottom limits are the set of coordinate plane bounding the first octant. The other limits are just the intersection of the equations in the previous set of limits. Thus, we are left simply with the sphere of radius 3 in the first octant, or $\rho = 3$. The bounds on φ, and θ in the first octant are $\theta = 0, \theta = \frac{\pi}{2}, \varphi = 0, \varphi = \frac{\pi}{2}$. The equation to be integrated can be converted to spherical algebraically. This leaves us with:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 \rho \cdot \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

Practice Problems.
Convert the following problems into the indicated coordinate system.

1. $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dzdydx$ into cylindrical
2. $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dzdydx$ into spherical [Hint: you will need two integrals in spherical as there will be a change in the outermost ρ equations.]
3. $\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\,dzdydx$ into spherical
4. $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} \cos(x^2+y^2)\,dzdydx$ into cylindrical
5. $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} \cos\left(\sqrt{x^2+y^2+z^2}\right)\,dzdydx$ into spherical