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Surface Integrals

Surface integrals are used for a number of basic things like calculating surface area, but they are also used for calculating things like the flow of material through a surface. This handout will lay out each type of problem in turn.

The simplest type of surface integral is $S = \int_R \int dS$. To calculate this integral, we need to break down what dS stands for. It's going to mean slightly different things in different cases depending on the kind of function we have. Let's begin with the case of an explicit function $z = g(x, y)$.

We need the normal vector to the surface. We can find that by finding $G(x, y, z) = z - g(x, y)$, which is just the above function solved for 0. Finding the gradient of this G function gives us the normal vector we need.

$$\nabla G = -g_x(x, y)\hat{\imath} - g_y(x, y)\hat{\jmath} + 1\hat{k}$$

dS then is given by $\|\nabla G\| dA$. So our surface integral becomes, in this instance $S = \int_R \int \sqrt{g_x^2 + g_y^2 + 1}\, dA$. The negative signs disappear when squared. The limits of integration are calculated from the extent of the area the surface is over, the projection of the surface onto the plane.

Because of the square root, it is by no means guaranteed that one will be able to calculate the surface area for a random function by hand, but we can for certain well-chosen functions.

Example 1. Calculate the surface area of the function $f(x, y) = xy$ over the region inside the circle of radius 4: $x^2 + y^2 \leq 16$.

First find the normal vector. $F(x, y, z) = z - xy \to \nabla F = -y\hat{\imath} - x\hat{\jmath} + 1\hat{k}$. Then our integral becomes $\int_R \int \sqrt{y^2 + x^2 + 1}\, dA$. Since the integral has the form of $x^2 + y^2$, as does our region, we will want to convert this integral into polar coordinates.

$$\int_0^{2\pi} \int_0^4 r\sqrt{r^2 + 1}\, dr d\theta$$

This integral can be integrated by substitution. Let $u = r^2 + 1, \frac{1}{2} du = r dr \to \int \frac{1}{2} u^{\frac{1}{2}} du = \frac{1}{3} u^{\frac{3}{2}}$. Thus $\frac{1}{3}(r^2 + 1)^{\frac{3}{2}}\Big|_0^4 = \frac{1}{3}\left(17^{\frac{3}{2}} - 1\right)$.

We can also calculate surface areas using parametric surfaces. Consider a function in parametric form. We also find dS by first finding its normal vector. However this process is different than for explicit functions.

Example 2. Find the surface area of the parametric surface $\vec{r}(u,v) = 4u\cos v\,\hat{\imath} + 4u\sin v\,\hat{\jmath} + u\hat{k}$, on the region bounded by $0 \le u \le 6, 0 \le v \le 2\pi$.

To find the normal vector we need to first find $\vec{r_u}$ *and* $\vec{r_v}$.

$$\vec{r_u} = 4\cos v\,\hat{\imath} + 4\sin v\,\hat{\jmath} + 1\hat{k}$$
$$\vec{r_v} = -4u\sin v\,\hat{\imath} + 4u\cos v\,\hat{\jmath} + 0\hat{k}$$

These vectors are on the surface, so to obtain the normal, we need the cross product.

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4\cos v & 4\sin v & 1 \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix} = -4u\cos v\,\hat{\imath} - 4u\sin v\,\hat{\jmath} + (16u\,cos^2 v + 16u\,sin^2 v)\hat{k}$$

This reduces to $-4u\cos v\,\hat{\imath} - 4u\sin v\,\hat{\jmath} + 16u\,\hat{k}$.

As with the normal vector for explicit functions, we need to integrate the magnitude of this vector.

$$S = \int_R \int \sqrt{16u^2 cos^2 v + 16u^2 sin^2 v + 256u^2}\,dudv = \int_R \int \sqrt{16u^2 + 256u^2}\,dudv$$
$$= \int_R \int \sqrt{272u^2}\,dudv = 4\int_0^{2\pi} \int_0^6 \sqrt{17}u\,dudv = 144\sqrt{17}\pi$$

Practice Problems.

Find the surface area of the given surfaces.

1. $f(x,y) = 12 + 2x - 3y, R: \{(x,y)|x^2 + y^2 \le 9\}$
2. $f(x,y) = 3 + x^{\frac{3}{2}}, R: rectangle\ with\ vertices\ (0,0), (0,4), (3,4), (3,0)$
3. $f(x,y) = \ln|sec\,x|, R: \left\{(x,y)|0 \le x \le \frac{\pi}{4}, 0 \le y \le \tan x\right\}$
4. $f(x,y) = \sqrt{x^2 + y^2}, R: \{(x,y)|x^2 + y^2 \le 9\}$
5. $\vec{r}(u,v) = 4u\hat{\imath} - v\hat{\jmath} + v\hat{k}, 0 \le u \le 2, 0 \le v \le 1$
6. $\vec{r}(u,v) = 2u\cos v\,\hat{\imath} + 2u\sin v\,\hat{\jmath} + u^2\hat{k}, 0 \le u \le 2, 0 \le v \le 2\pi$
7. $\vec{r}(u,v) = \sin u\cos v\,\hat{\imath} + u\hat{\jmath} + \sin u\sin v\,\hat{k}, 0 \le u \le \pi, 0 \le v \le 2\pi$

Now let's consider a more general type of surface integral problem. Suppose, for instance, that we wished to calculate the mass of a lamina, but instead of being in the plane, it's stretched out over some surface. We can calculate the mass by the integral $\int_R \int f(x,y)dS$ *or* $\int_R \int f(x,y,z)dS$ where f(x,y) or f(x,y,z) represents the density function for the mass on the surface. We will have to deal with the function, in addition to calculating dS, but most of our procedures will remain the same.

Example 3. Set up the double integral needed to calculate the surface integral $\iint_S x^2 - 2xy\, dS, on\ the\ surface\ S: z = 10 - x^2 - y^2, 0 \leq x \leq 2, 0 \leq y \leq 2$

Since we have a function of (x,y) already, we don't need to do anything about that, but we do need to work on the dS. Like before, we need the normal to the surface.

$$G(x, y, z) = z - 10 + x^2 + y^2 \rightarrow \nabla G = 2x\hat{\imath} + 2y\hat{\jmath} + 1\hat{k}$$

So the surface integral becomes: $\int_0^2 \int_0^2 (x^2 - 2xy)\sqrt{4x^2 + 4y^2 + 1}\, dydx$

Switching this integral to polar isn't going to be useful. The second term can be done by substitution, but the first term will have to be done by trig substitution. Even better, would be to let Mathematica take care of it for us!

Example 4. Calculate the surface integral $\int_S \int \frac{xy}{z} dS$, on the surface $S: z = x^2 + y^2, 4 \leq x^2 + y^2 \leq 16$.

We calculate the normal vector first. $G(x, y, z) = z - x^2 - y^2 \rightarrow \nabla G = -2x\hat{\imath} - 2y\hat{\jmath} + 1\hat{k}$. Because our function has a z in it, we will need to substitute using the equation of the surface provided.

$$\int_S \int \frac{xy}{x^2 + y^2} \sqrt{4x^2 + 4y^2 + 1} dA$$

Since we are integrating over an annulus with inner radius 2, and outer radius 4, we will convert everything to polar coordinates.

$$\int_0^{2\pi} \int_2^4 \frac{r^2 \sin\theta \cos\theta}{r^2} \sqrt{4r^2 + 1}\, rdrd\theta = \int_0^{2\pi} \int_2^4 \sin\theta \cos\theta \sqrt{4r^2 + 1}\, rdrd\theta$$

This can easily be completed by substitution. For in the inside integral, let $u = 4r^2 + 1, \frac{1}{8}du = rdr \rightarrow \int \frac{1}{8} u^{\frac{1}{2}} du = \frac{1}{12} u^{\frac{3}{2}} \rightarrow \frac{1}{12}(4r^2 + 1)^{\frac{3}{2}}\Big|_2^4 = \frac{1}{12}\left(65^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)$. This leaves us with:

$$\frac{1}{12}\left(65^{\frac{3}{2}} - 17^{\frac{3}{2}}\right) \int_0^{2\pi} \sin\theta \cos\theta\, d\theta$$

This integral also needs to be done by substitution, with $u = \sin\theta, du = \cos\theta\, d\theta$.

$$\frac{1}{12}\left(65^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)\frac{1}{2}sin^2\theta\Big|_0^{2\pi} = 0$$

How are we to interpret this zero? Difficult to do with mass since mass is never negative, but we can make sense of this if our original function represented charge density. This value could represent the net charge on the surface, implying that there are equal regions of net negative charge and net positive charge which cancel each other out.

Practice problems.

Calculate the surface integral $\iint_S f(x,y,z)dS$ of the given function over the given surface. Note: if the surface is parameterized, you will need to make the substitutions for x and y in order to integrate.

8. $f(x,y,z) = x - 2y + z, S: z = \frac{2}{3}x^{\frac{3}{2}}, 0 \le x \le 1, 0 \le y \le x$
9. $f(x,y) = xy, S: z = \frac{1}{2}xy, 0 \le x \le 2, 0 \le y \le \sqrt{4 - x^2}$
10. $f(x,y) = x + y, S: \vec{r}(u,v) = 2\cos u\,\hat{\imath} + 2\sin u\,\hat{\jmath} + v\hat{k}, 0 \le u \le \frac{\pi}{2}, 0 \le v \le 1$
11. $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}, S: z = \sqrt{x^2 + y^2}, (x-1)^2 + y^2 \le 1$

To calculate the flow through a surface, we have another type of surface integral involving the vector field equation that defines the flow. This integral is given by:

$$\iint_S \vec{F} \cdot \vec{N} dS$$

Where $\vec{N}$ is the unit normal to the surface. However, since the unit normal is just ∇G turned into a unit vector, this expression partially cancels with our expression for dS, reducing the integral to:

$$\iint_S \vec{F} \cdot \vec{N} dS = \iint_R \vec{F} \cdot \overrightarrow{\nabla G} dA = \iint_S \vec{F} \cdot (\vec{r_u} \times \vec{r_v}) dS$$

Since the dot product will produce a function rather than a vector, we just integrate the resulting function over the area under consideration. To get the sign correct for the flow, we must be using the upward normal to the surface. Eventually, we will be doing enclosed regions and so “upward” may be replaced with “outward”.

Example 5. Calculate $\iint_S \vec{F} \cdot \vec{N} dS$ for the vector field $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, through the surface $x^2 + y^2 + z^2 = 36$ in the first octant.

Solving for z, we get $z = \sqrt{36 - x^2 - y^2} \rightarrow G(x, y, z) = z - \sqrt{36 - x^2 - y^2} \rightarrow \nabla G = \frac{x}{\sqrt{36-x^2-y^2}}\hat{\imath} + \frac{y}{\sqrt{36-x^2-y^2}}\hat{\jmath} + 1\hat{k}$.

$$\vec{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{x^2 + y^2}{\sqrt{36 - x^2 - y^2}} + \sqrt{36 - x^2 - y^2}$$
$$= \frac{x^2 + y^2 + 36 - x^2 - y^2}{\sqrt{36 - x^2 - y^2}} = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

We should convert this to polar for integration.

$$\int_0^{\frac{\pi}{2}} \int_0^6 \frac{36 r dr d\theta}{\sqrt{36 - r^2}}$$

This integral can be done easily with substitution: let $u = 36 - r^2, -\frac{1}{2}du = rdr \rightarrow \int 18u^{-\frac{1}{2}}du = -36u^{\frac{1}{2}} \rightarrow -36\sqrt{36 - r^2}\Big|_0^6 = -36(0 - 6) = 216$.

$$\int_0^{\frac{\pi}{2}} 216 d\theta = 108\pi$$

Practice Problems.

Calculate $\iint_S \vec{F} \cdot \vec{N} dS$ for the given vector field and the given surface.

12. $\vec{F}(x, y, z) = x\hat{\imath} + y\hat{\jmath}, S: z = 6 - 3x - 2y, first\ octant$
13. $\vec{F}(x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, S: z = 1 - x^2 - y^2, z \leq 0$

Sometimes we are interested in calculating the flux over a closed surface. This problem will be very much like Example 5, but we will have to do a surface integral for *each* surface face bounding the volume, and then add the results together. We will also need to be careful to properly orient the normal vectors as outward normals.

Example 6. Calculate the flux for the vector field $\vec{F}(x, y, z) = 4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}$ on the unit cube bounded by x=0, x=1, y=0, y=1, z=0, z=1.

We have six surfaces to deal with: 1) x=0 is the back face on the yz-plane, and the normal pointing outward from the interior is $-\hat{\imath}$; 2) x=1 is the front face, and the normal outward from the interior is $\hat{\imath}$; 3) y=0 is the left face on the xz-plane and the outward normal is $-\hat{\jmath}$; 4) y=1 is the right face and the outward normal is $\hat{\jmath}$; 5) z=0 is the bottom face on the xy-plane and the outward normal is $-\hat{k}$; 6) z=1 is the top face and the outward normal is $\hat{k}$.

Constructing our 6 integrals then:

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot -\hat{\imath}dS = \iint_S -4xy\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot \hat{\imath}dS = \iint_S 4xy\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot -\hat{\jmath}dS = \iint_S -z^2\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot \hat{\jmath}dS = \iint_S z^2\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot -\hat{k}dS = \iint_S -yz\,dA$$

$$\iint_S \vec{F}\cdot\vec{N}dS = \iint_S \left(4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}\right)\cdot \hat{k}dS = \iint_S yz\,dA$$

The first two integrals are projected onto the yz-plane as a square between zero and one in both dimensions, and so are dydz integrals. We can replace x with x=0 and x=1 respectively.

$$\iint_S -4xy\,dA = \int_0^1\int_0^1 -4(0)ydydz = 0$$

$$\iint_S 4xy\,dA = \int_0^1\int_0^1 4(1)ydydz = \int_0^1\int_0^1 4ydydz = \int_0^1 2y^2|_0^1dz = \int_0^1 2dz = 2$$

The second two integrals project onto the xz-plane as a square between zero and one and so are integrals in dxdz. We replace any y's with y=0 and y=1 respectively (though, here, there are no y's).

$$\iint_S -z^2\,dA = \int_0^1\int_0^1 -z^2dzdx = \int_0^1 -\frac{1}{3}z^3\Big|_0^1 dx = \int_0^1 -\frac{1}{3}dx = -\frac{1}{3}$$

$$\iint_S z^2\,dA = \int_0^1\int_0^1 z^2dzdx = \int_0^1 \frac{1}{3}z^3\Big|_0^1 dx = \int_0^1 \frac{1}{3}dx = \frac{1}{3}$$

These two will cancel out when added.

Finally, we do the last two, which are protected onto the xy-plane, both variables between zero and one, and so are integrals in dydx. We replace z=0 and z=1 into the integrals respectively.

$$\iint_S -yz\,dA = \int_0^1\int_0^1 -y(0)dxdy = 0$$

$$\iint_S yz\,dA = \int_0^1\int_0^1 y(1)dxdy = \int_0^1\int_0^1 ydxdy = \int_0^1 \frac{1}{2}y^2\Big|_0^1 dx = \int_0^1 \frac{1}{2}dx = 1/2$$

At last, we add all the surface integrals together to obtain $2+\frac{1}{2}=\frac{5}{2}$.

At this point it's useful to note some terminology here. A closed surface integral like this that produces a net positive flow is called a *source*. It means that there is more coming out the surface than is going in.

A closed surface integral that gives a negative value is called a *sink*, meaning that more material is flowing into the surface than is coming out.

A closed surface integral that comes out to be zero is called an *incompressible flow*, meaning that the same amount that is going in is also coming out.

Practice Problems.
Calculate the flow through the closed surface.

14. $\vec{F}(x,y,z) = (x+y)\hat{\imath} + y\hat{\jmath} + z\hat{k}, S: z = 16 - x^2 - y^2, z = 0$
15. $\vec{F}(x,y,z) = 2x\hat{\imath} - 2y\hat{\jmath} + z^2\hat{k}, S: cylinder\ x^2 + y^2 = 4, 0 \le z \le 5$
16. $\vec{F}(x,y,z) = (2x-y)\hat{\imath} + (z-2y)\hat{\jmath} + z\hat{k}, S: plane\ 2x + 4y + 2z = 12, z = 0, x = 0, y = 0$

Closed surfaces can be a lot of work, particularly if they have a lot of surfaces bounding them. It would be nice if we had a way to calculate a closed surface in a single integral instead of many. This is what the Divergence Theorem provides us. It says that for a closed surface the surface integral $\iint_S \vec{F}\cdot\vec{N}dS = \iiint_V \nabla\cdot\vec{F}dV$, which is the divergence of the field, integrated over the volume bounded by the surfaces.

Example 7. Recalculate Example 6 using the Divergence Theorem and verify the results.

First, we need to calculate the divergence of the vector field $\vec{F}(x,y,z) = 4xy\hat{\imath} + z^2\hat{\jmath} + yz\hat{k}$. This is $\nabla\cdot\vec{F} = 4y + 0 + y = 5y$. The volume is just a cube, with all variables between 0 and 1. Thus, our triple integral becomes:

$$\int_0^1 \int_0^1 \int_0^1 5y \, dz dy dx = \frac{5}{2}$$

Practice Problems.

Find the flux of the given vector field through the closed surface.

17. Redo the previous three problems (14-16) using the Divergence Theorem and verify your results.
18. $\vec{F}(x,y,z) = xe^z\hat{\imath} + ye^z\hat{\jmath} + e^z\hat{k}, S: z = 4 = y, z = 0, x = 0, x = 6, y = 0$

Surface integrals can also be used to evaluate line integrals along the edge of a surface. This subject and Stokes' Theorem is treated at the end of Line Integrals handout.