Tangents & Normals Key

1. $\vec r(t) = 6\cos t\,\hat i + 2\sin t\,\hat j \qquad t = \pi/3 \qquad \sin \pi/3 = \frac{\sqrt3}{2} \quad \cos \pi/3 = 1/2$

$\vec r\,'(t) = -6\sin t\,\hat i + 2\cos t\,\hat j$

$\|\vec r\,'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t} = \sqrt{4\sin^2 t + 4\cos^2 t + 32\sin^2 t} = \sqrt{4 + 32\sin^2 t} = 2\sqrt{1+8\sin^2 t}$

$\vec T(t) = \dfrac{-6\sin t\,\hat i + 2\cos t\,\hat j}{2\sqrt{1+8\sin^2 t}} = \dfrac{-3\sin t\,\hat i + \cos t\,\hat j}{\sqrt{1+8\sin^2 t}} \Rightarrow \vec T(\pi/3) = \dfrac{-3(\frac34)\hat i + \frac12\hat j}{\sqrt{1+8(\frac34)}} =$

$\dfrac{-\frac94\hat i + \frac12\hat j}{\sqrt{1+6}} = -\frac{9}{4\sqrt7}\hat i + \frac{1}{2\sqrt7}\hat j$

$\vec N(t) = \dfrac{T'(t)}{\|T'(t)\|} \qquad T'(t) = \dfrac{-3\cos t\,\hat i - \sin t\,\hat j}{\sqrt{1+8\sin^2 t}} + \dfrac{-3\sin t\,\hat i + \cos t\,\hat j}{\left(\sqrt{1+8\sin^2 t}\right)^3}(-\tfrac12)(8\sin t\cos t \cdot 2)$

$= \dfrac{(-3\cos t\,\hat i - \sin t\,\hat j)(1+8\sin^2 t) + (-3\sin t\,\hat i + \cos t\,\hat j)(-8\sin t\cos t)}{(1+8\sin^2 t)^{3/2}}$

$= \dfrac{(-3\cos t - 24\cos t\sin^2 t + 24\sin^2 t\cos t)\hat i + (-\sin t - 8\sin^3 t - 8\cos^2 t\sin t)\hat j}{\left(\sqrt{1+8\sin^2 t}\right)^3}$

$= \dfrac{-3\cos t\,\hat i + -\sin t\,\hat j\,(1+8\sin^2 t + 8\cos^2 t)}{(1+8\sin^2 t)^{3/2}} = \dfrac{-3\cos t\,\hat i - 9\sin t\,\hat j}{(1+8\sin^2 t)^{3/2}}$

$\|T'(t)\| = \dfrac{1}{(1+8\sin^2 t)^{3/2}}\sqrt{9\cos^2 t + 81\sin^2 t} = \dfrac{3\sqrt{1+8\sin^2 t}}{(1+8\sin^2 t)^{3/2}}$

$\vec N(t) = \dfrac{-3\cos t\,\hat i - 9\sin t\,\hat j}{3\sqrt{1+8\sin^2 t}} = \dfrac{-\cos t\,\hat i - 3\sin t\,\hat j}{\sqrt{1+8\sin^2 t}} \qquad N(\tfrac{\pi}{3}) = \frac{-1}{2\sqrt7}\hat i - \frac{9}{4\sqrt7}\hat j$

$\vec B(t) = T \times N = \dfrac{1}{1+8\sin^2 t}\begin{vmatrix} \hat i & \hat j & \hat k \\ -3\sin t & \cos t & 0 \\ -\cos t & -3\sin t & 0 \end{vmatrix} = 0\hat i + 0\hat j + \dfrac{(9\sin^2 t + \cos^2 t)\hat k}{} = \dfrac{1+8\sin^2 t\,\hat k}{1+8\sin^2 t} = \hat k$

2. $\vec{r}(t) = 2\sin t\,\hat{i} + 2\cos t\,\hat{j} + 4\hat{k}$ $\quad P(\sqrt{2}, \sqrt{2}, 4)$ $\quad t = \pi/4$

$r'(t) = 2\cos t\,\hat{i} - 2\sin t\,\hat{j}$ $\quad \|\vec{r}'(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t} = \sqrt{4} = 2$

$T(t) = \cos t\,\hat{i} - \sin t\,\hat{j}$ $\quad T(\pi/4) = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$N(t) = -\sin t\,\hat{i} - \cos t\,\hat{j}$ $\quad N(\pi/4) = -\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$$\vec{B} = T \times N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -\sin t & -\cos t & 0 \end{vmatrix} = -\hat{k}$$

3. $\vec{r}(t) = \ln t\,\hat{i} + (t+1)\hat{j}$ $\quad t = 2$

$r'(t) = \frac{1}{t}\hat{i} + 1\hat{j}$ $\quad \|\vec{r}'(t)\| = \sqrt{\frac{1}{t^2} + 1} = \sqrt{\frac{1+t^2}{t^2}} = \frac{\sqrt{1+t^2}}{t}$

$T(t) = \left(\frac{1}{t}\hat{i} + 1\hat{j}\right)\frac{t}{\sqrt{1+t^2}} = \frac{\hat{i} + t\hat{j}}{\sqrt{1+t^2}}$ $\quad \vec{T}(2) = \frac{\hat{i}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\hat{j}$

$$T'(t) \quad \frac{\hat{j}}{\sqrt{1+t^2}} + \frac{\hat{i}+t\hat{j}}{(1+t^2)^{3/2}}(-1/2)(2t) = \frac{-t\hat{i} + [(1+t^2) - t^2]\hat{j}}{(1+t^2)^{3/2}} =$$

$$\frac{-t\hat{i} + \hat{j}}{(1+t^2)^{3/2}} \qquad \|T'(t)\| = \frac{1}{(1+t^2)^{3/2}}\sqrt{t^2+1}$$

$$N(t) = \frac{-t\hat{i} + \hat{j}}{(1+t^2)^{3/2}} \cdot \frac{(t^2+1)^{3/2}}{\sqrt{t^2+1}} = \frac{-t\hat{i} + \hat{j}}{\sqrt{t^2+1}} \qquad N(2) = \frac{-2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$$

$$B(t) = \frac{1}{1+t^2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & 0 \\ -t & 1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \frac{(1+t^2)\hat{k}}{1+t^2} = \hat{k}$$

4. $\vec{r}(t) = (t^3 - 4t)\hat{i} + 2t^2\hat{j}$ $\quad t = 1$

$r'(t) = (3t^2 - 4)\hat{i} + 4t\hat{j}$ $\quad \|\vec{r}'(t)\| = \sqrt{(3t^2-4)^2 + (4t)^2} =$

$$\sqrt{9t^4 - 24t^2 + 16 + t^2} = \sqrt{9t^4 - 8t^2 + 16}$$

$$T(t) = \frac{(3t^2-4)\hat{i} + 4t\hat{j}}{\sqrt{9t^4 - 8t^2 + 16}} \qquad \vec{T}(1) = \frac{-\hat{i} + 4\hat{j}}{\sqrt{17}} = -\frac{1}{\sqrt{17}}\hat{i} + \frac{4}{\sqrt{17}}\hat{j}$$

4 cont'd.

$$T'(t) = \frac{(6t)\hat{i} + 4\hat{j}}{\sqrt{9t^4-8t^2+16}} + \frac{(3t^2-4)\hat{i} + 4t\hat{j}}{(9t^4-8t^2+16)^{3/2}}(-1/2)(36t^3-16t)$$

$$\frac{[6t(9t^4-8t^2+16) - (3t^2-4)(18t^3-8t)]\hat{i} + [4(9t^4-8t^2+16) - 4t(18t^3-8t)]\hat{j}}{(9t^4-8t^2+16)^{3/2}}$$

$$= \frac{[54t^5 - 48t^3 + 96t - (54t^5 - 24t^3 - 72t^3 + 32t)]\hat{i} + [36t^4 - 32t^2 + 64 - 72t^4 + 32t^2]\hat{j}}{(9t^4-8t^2+16)^{3/2}}$$

$$= \frac{[48t^3 + 64t]\hat{i} + [-36t^4 + \;\; + 64]\hat{j}}{(9t^4-8t^2+16)^{3/2}} =$$

$$64 - 36t^4 = (8-6t^2)(8+6t^2) = 4(4-3t^2)(4+3t^2)$$

$$48t^3 + 64t = 8t(6t^2+8) = 16t(3t^2+4)$$

$$\frac{4(3t^2+4)[4t\hat{i} + (4-3t^2)\hat{j}]}{(9t^4-8t^2+16)^{3/2}}$$

$$\|T'(t)\| = \frac{4(3t^2+4)}{(9t^4-8t^2+16)^{3/2}}\sqrt{(4t)^2 + (4-3t^2)^2} = \frac{4(3t^2+4)}{(9t^4-8t^2+16)^{3/2}}\sqrt{9t^4-8t^2+16}$$

$$N(t) = \frac{4(3t^2+4)[4t\hat{i} + (4-3t^2)\hat{j}]}{(9t^4-8t^2+16)^{3/2}} \cdot \frac{(9t^4-8t^2+16)^{3/2}}{4(3t^2+4)\sqrt{9t^4-8t^2+16}}$$

$$= \frac{4t\hat{i} + (4-3t^2)\hat{j}}{\sqrt{9t^4-8t^2+16}} \qquad N(1) = \frac{4}{\sqrt{17}}\hat{i} + \frac{1}{\sqrt{17}}\hat{j}$$

4 contd

$$\vec{B} = \frac{1}{9t^4-8t^2+16}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2-4 & 4t & 0 \\ 4t & 4-3t^2 & 0 \end{vmatrix} = \frac{9t^4-8t^2+16}{9t^4-8t^2+16}\hat{k} = \hat{k}$$

5. $\vec{r}(t) = 4t\hat{i} - 4t\hat{j} + 2t\hat{k} \quad t=2$

$\vec{r}\,'(t) = 4\hat{i} - 4\hat{j} + 2\hat{k} \qquad \|r'(t)\| = \sqrt{16+16+4} = \sqrt{36} = 6$

$T(t) = \frac{4}{6}\hat{i} - \frac{4}{6}\hat{j} + \frac{2}{6}\hat{k} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

$N(t) = \vec{0}$ since $T'(t) = \vec{0}$ There is no normal vector

6. $F(x,y,z) = x^2+4y^2+z^2-36=0 \quad P(2,-2,4)$

$\nabla F = \langle 2x, 8y, 2z\rangle \qquad \nabla F(2,-2,4) = \langle 4,-16,8\rangle$

tangent plane: $4(x-2)-16(y+2)+8(z-4)=0$

normal line: $\frac{x-2}{4} = \frac{y+2}{-16} = \frac{z-4}{8}$

7. $F(x,y,z) = x^2+y^2+z-9=0 \quad P(1,2,4)$

$\nabla F = \langle 2x, 2y, 1\rangle \qquad \nabla F(1,2,4) = \langle 2,4,1\rangle$

tangent plane: $2(x-1)+4(y-2)+(z-4)=0$

normal line: $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}$

8. $F(x,y,z) = y\ln xz^2 - 2 = y[\ln x + 2\ln z] - 2 = 0 \quad P(e,2,1)$

$\nabla F = \langle y/x, \ln x + 2\ln z, \frac{2y}{z}\rangle \quad \nabla F(e,2,1) = \langle 2/e, 1, 4\rangle$

tangent plane: $\frac{2}{e}(x-e)+(y-2)+4(z-1)=0$

normal line: $\frac{x-e}{\left(\frac{2}{e}\right)} = \frac{y-2}{1} = \frac{z-1}{4}$

9. $F(x,y,z) = xyz - 10$ $\quad P(1,2,5)$

$\nabla F = \langle yz, xz, xy \rangle$ $\quad \nabla F(1,2,5) = \langle 10, 5, 2 \rangle$

tangent plane: $10(x-1) + 5(y-2) + 2(z-5) = 0$

normal line: $\dfrac{x-1}{10} = \dfrac{y-2}{5} = \dfrac{z-5}{2}$

10. $F(x,y,z) = 2xy - z^3$ $\quad P(2,2,2)$

$\nabla F = \langle 2y, 2x, -3z^2 \rangle$ $\quad \nabla F(2,2,2) = \langle 4, 4, -12 \rangle$

tangent plane: $4(x-2) + 4(y-2) - 12(z-2) = 0$

normal line: $\dfrac{x-2}{4} = \dfrac{y-2}{4} = \dfrac{z-2}{-12}$

11. to be horizontal the normal vector ∇F must be $\pm\hat{k}$ or some constant multiple of it. therefore $\frac{\partial F}{\partial x}$ & $\frac{\partial F}{\partial y}$ must $= 0$.

$\nabla F = \langle 6x-3, 4y+4, -1 \rangle$

$6x - 3 = 0 \Rightarrow x = 1/2 \qquad 4y + 4 = 0 \Rightarrow y = -1$

must be horizontal at the point $(1/2, -1, z)$ on the graph.

12. $\vec{r}(u,v) = u\hat{\imath} + v\hat{\jmath} + uv\hat{k}$ $\quad P(1,1,1)$ $\quad u=1, v=1$

$\vec{r}_u = \hat{\imath} + v\hat{k}$ $\quad \vec{r}_v = \hat{\jmath} + u\hat{k}$ $\quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} =$

$(0-v)\hat{\imath} - (u-0)\hat{\jmath} + \hat{k} = -v\hat{\imath} - u\hat{\jmath} + \hat{k} = \vec{N}$

$\vec{N}(1,1) = -\hat{\imath} - \hat{\jmath} + \hat{k} = \langle -1, -1, 1 \rangle$

tangent plane: $-(x-1) - (y-1) + (z-1) = 0$

normal line: $\vec{r}(t) = (-t+1)\hat{\imath} + (-t+1)\hat{\jmath} + (t+1)\hat{k}$

13. $\vec{r}(u,v) = 2\cos u\,\hat{i} + v\hat{j} + 2\sin u\,\hat{k} \quad P(2,4,0)$ ⑥

$\vec{r}_u = -2\sin u\,\hat{i} + 2\cos u\,\hat{k} \quad \vec{r}_v = \hat{j} \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin u & 0 & 2\cos u \\ 0 & 1 & 0 \end{vmatrix}$

$= (0 - 2\cos u)\hat{i} - (0)\hat{j} + (-2\sin u - 0)\hat{k}$

$\vec{N} = -2\cos u\,\hat{i} - 2\sin u\,\hat{k} \qquad u=0,\ v=4$

$N(0,4) = -2\hat{i} - 0\hat{k}$

tangent plane: $-2(x-2) = 0$

normal line: $(-2t+2)\hat{i} + 4\hat{j} + 0\hat{k} = \vec{r}(t)$

14. $\vec{r}(u,v) = 3\cos v\cos u\,\hat{i} + 3\cos v\sin u\,\hat{j} + 5\sin v\,\hat{k} \quad P(3,0,0)$

$\vec{r}_u = -3\cos v\sin u\,\hat{i} + 3\cos v\cos u\,\hat{j} + 0\hat{k} \qquad v=0$

$\vec{r}_v = -3\sin v\cos u\,\hat{i} - 3\sin v\sin u\,\hat{j} + 5\cos v\,\hat{k} \qquad u=0$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\cos v\sin u & 3\cos v\cos u & 0 \\ -3\sin v\cos u & -3\sin v\sin u & 5\cos v \end{vmatrix} =$

$(15\cos^2 v\cos u - 0)\hat{i} - (-15\cos^2 v\sin u - 0)\hat{j} + (9\cos v\sin v\sin^2 u + 9\cos v\sin v\cos^2 u)\hat{k}$

$= 15\cos^2 v\cos u\,\hat{i} + 15\cos^2 v\sin u\,\hat{j} + 9\cos v\sin v\,\hat{k}$

$N(0,0) = 15\hat{i} + 0\hat{j} + 0\hat{k}$

tangent plane: $15(x-3) = 0$

normal line: $(15t+3)\hat{i} + 0\hat{j} + 0\hat{k} = \vec{r}(t)$

15. $\vec{r}(u,v) = u\hat{i} + \frac{1}{4}v^3\hat{j} + v\hat{k} \quad P(-1,2,2) \qquad u=-1,\ v=2$

$\vec{r}_u = \hat{i} \quad \vec{r}_v = \frac{3}{4}v^2\hat{j} + \hat{k} \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 3/4v^2 & 1 \end{vmatrix} = 0\hat{i} - 1\hat{j} + \frac{3}{4}v^2\hat{k}$

$N(-1,2) = \langle 0, -1, 3 \rangle$

tangent plane: $-1(y-2) + 3(z-2) = 0$

normal line: $-1\hat{i} + (-t+2)\hat{j} + (3t+2)\hat{k} = \vec{r}(t)$

16. $F(x,y,z) = x^2+y^2+z^2-36=0$

$\nabla F = \langle 2x, 2y, 2z\rangle$ outward normal

$\langle -2x, -2y, -2z\rangle$ inward normal

$\|\nabla F\| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{36} = 12$

unit normal: $\dfrac{x\hat{\imath}+y\hat{\imath}+z\hat{\jmath}}{\sqrt{6}}$ Sphere

17. Same as above but add vectors for planes. inward to volume is $\hat{\imath}, \hat{\jmath}, \hat{k}$ & outward $-\hat{\imath}, -\hat{\jmath}, -\hat{k}$

18. $F(x,y,z) = z-1+x^2+y^2$

$\nabla F = \langle 2x, 2y, 1\rangle$ outward normal downward opening paraboloid

$\langle -2x, -2y, -1\rangle$ inward normal

$\|\nabla F\| = \sqrt{1+4x^2+4y^2}$

unit normal: $\dfrac{2x\hat{\imath}+2y\hat{\jmath}+1\hat{k}}{\sqrt{1+4x^2+4y^2}}$

19. $F(x,y,z) = x^2+y^2-z$ paraboloid/opening up

$\nabla F = \langle 2x, 2y, -1\rangle$ outward normal alone (& w/other shapes

$\langle -2x, -2y, 1\rangle$ upward/inward normal) these reverse (w/the plane below)

unit normal: $\dfrac{2x\hat{\imath}+2y\hat{\imath}+\hat{k}}{\sqrt{1+4x^2+4y^2}}$

$G(x,y,z) = x^2+y^2-4$ $\nabla G = \langle 2x, 2y, 0\rangle$ outward normal (cylinder)

$\langle -2x, -2y, 0\rangle$ inward normal

unit normal $\dfrac{x\hat{\imath}+y\hat{\imath}}{\sqrt{x^2+y^2}} = \dfrac{x}{2}\hat{\imath} + \dfrac{y}{2}\hat{\jmath}$

$z=-1$ $F(x,y,z) = z+1$ $\nabla F = \langle 0,0,1\rangle$ Since this is a plane any normal vector could be inward or outward

in this combo, though $\hat{k}$ is the inward normal & $-\hat{k}$ the outward one

20. $F(x,y,z) = 6-3x-2y-z$ first octant

$\nabla F = \langle -3,-2,-1\rangle$ this is the inward normal for the volume bound by the coordinate planes w/ $\langle 3,2,1\rangle$ the outward normal

the planes themselves are inward w/ $\hat{j}$, $\hat{k}$ & $\hat{i}$ or outward $-\hat{j}$, $-\hat{k}$, $-\hat{i}$

the unit normal for the plane is $\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \rangle$

$\|\nabla F\| = \sqrt{9+4+1} = \sqrt{14}$