Vector Fields & Potential Functions

A vector field is best exemplified by a “force field”. In each location in the plane, or in 3-space, the field applies a force in a particular direction, and the strength and direction of that field changes depending on where one is in the place or the space. A gravitational field is a relatively simple example, at least when there is only one dominant body, but electrical fields are another example that can easily be more complex with the introduction of dipoles and combinations of charged particles. We will be looking at generic fields that don’t necessarily relate to specific examples, but what is covered here, can be applied to all these real-world examples.

We first want to be able to graph a vector field so we can see what is going on. We will concentrate on examples in two dimensions since they are easiest to draw on paper, but what we say can be extended to three dimensions, and we will show an image, drawn by a computer program, of what a three dimensional field might look like.

Example 1. Let’s consider the vector field given by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$. This field is defined by the coordinates of the point in the plane. For instance, if the point in the plane is (1,2), then the vector in the field points in the direction <-2,1>. We need to plot a fairly large number of vectors to get an idea of what the field is doing overall. We also will not be able to plot the vectors at their full length because then they will likely overlap with one another, making the field difficult to see. We will therefore scale them by a constant factor, let’s say by 1/3 (if your vectors are longer, you will need a smaller scale factor so the vectors don’t run into each other). This will still give us the direction and relative magnitude that we need.

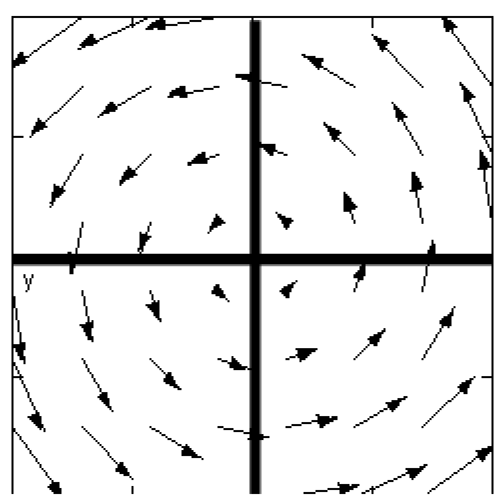
Begin by calculating the vectors that we need to plot the graph. We will need to specify a point where the vector begins, the value of the vector at that point, the magnitude of the vector, and the magnitude of the vector we will plot. We will need a minimum of 10 well-chosen vectors (i.e. not all on one line, or only on the axes), but for completeness, I will calculate more like 20ish.

Point	**Vector**	**Magnitude**	**Scaled Length**
(0,0)	<0,0>	0	0
(1,0)	<0,1>	1	1/3
(0,1)	<-1,0>	1	1/3
(-1,0)	<0,-1>	1	1/3
(0,-1)	<1,0>	1	1/3
(1,1)	<-1,1>	$\sqrt{2}$	$\frac{\sqrt{2}}{3} \approx .47$
(1,-1)	<1,1>	$\sqrt{2}$	$\frac{\sqrt{2}}{3} \approx .47$
(-1,1)	<-1,-1>	$\sqrt{2}$	$\frac{\sqrt{2}}{3} \approx .47$
(-1,-1)	<1,-1>	$\sqrt{2}$	$\frac{\sqrt{2}}{3} \approx .47$

(2,0)	<0,2>	2	2/3
(-2,0)	<0,-2>	2	2/3
(0,2)	<-2,0>	2	2/3
(0,-2)	<2,0>	2	2/3
(2,1)	<-1,2>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(-2,1)	<-1,-2>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(2,-1)	<1,2>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(-2,-1)	<1,-2>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(1,2)	<-2,1>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(1,-2)	<2,1>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(-1,2)	<-2,-1>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(-1,-2)	<2,-1>	$\sqrt{5}$	$\frac{\sqrt{5}}{3} \approx .75$
(2,2)	<-2,2>	$2\sqrt{2}$	$\frac{2\sqrt{2}}{3} \approx .94$
(-2,2)	<-2,-2>	$2\sqrt{2}$	$\frac{2\sqrt{2}}{3} \approx .94$
(2,-2)	<2,2>	$2\sqrt{2}$	$\frac{2\sqrt{2}}{3} \approx .94$
(-2,-2)	<2,-2>	$2\sqrt{2}$	$\frac{2\sqrt{2}}{3} \approx .94$

Remember to plot each vector from the point that defines it, in the direction defined by the vector, and with magnitude defined by the scaled length. When doing this by hand, don't worry about being precise. We are looking for general properties, so eyeballing it is fine, but do aim for consistency.

Our final plot looks like this:

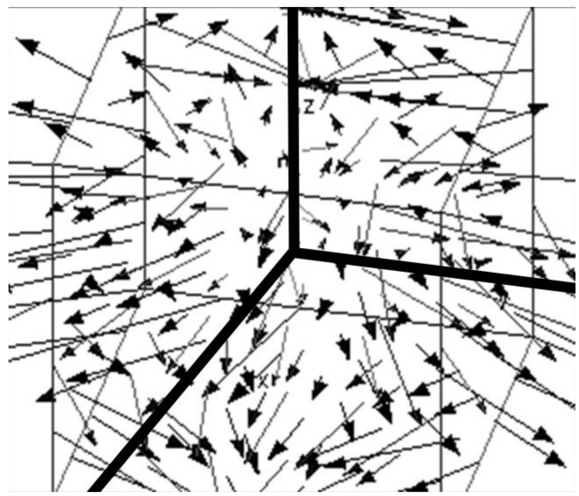


You see how as you go away from the origin, the vectors get bigger? And in the center is what we call a stationary point, where nothing moves.

Plotting vector fields can seem tedious at first, but you'll notice that you can often establish a pattern of points where the magnitudes will all be the same. This will reduce your calculations, if you can be systematic in this way. You will also often find that a pattern develops rather quickly. Calculate a set of vectors in one quadrant, and then just make a transformation into the others.

Example 2. Let's look at a three-dimensional example: $\vec{F}(x,y,z) = x^2\vec{i} + xyz\vec{j} + z\vec{k}$. The vector field graph was obtained from a Java Applet posted on the Saint Louis University website.

I chose this one to look a little crazier than the other one, but the procedure is essentially the same. Study the graph. Can you see any general trends in vector field? If you put a particle in the field and let the field apply force to it, where will the particle end up? Does it matter where it begins?



Practice Problems. Plot the 2-dimensional vector fields below. Describe in words the shape of the field; what will happen to a particle placed in the field? For the fields with denominators, you may wish to scale the vectors up instead of down (or not all all).

a. $\vec{F}(x,y) = x\vec{i} + y\vec{j}$

b. $\vec{F}(x,y) = x^2\vec{i} + xy\vec{j}$

c. $\vec{F}(x,y) = (x-1)\vec{i} - (y+2)\vec{j}$

d. $\vec{F}(x,y) = (x^2+y^2)\vec{i} - xy\vec{j}$

e. $\vec{F}(x,y) = \dfrac{1}{(x^2+y^2)}\vec{i} + \dfrac{1}{(x^2+y^2)}\vec{j}$

f. $\vec{F}(x,y) = \dfrac{x}{(x^2-y^2)}\vec{i} - \dfrac{y}{(x^2-y^2)}\vec{j}$

Certain types of vector fields have special properties that are important for solving problems. One of these special properties if the vector field is the gradient of a function, called a potential function. If such a function exists, the field is called conservative. Many real-world problems have such potential functions. Let's first consider an example where we know the potential function in advance, and then how to calculate it from the field in both two and three dimensions.

Example 3. Find the conservative vector field associated with the potential function $f(x,y) = \ln(x^2 y)$.

We need to calculate the gradient of the function.

$$\nabla f(x,y) = \frac{\partial}{\partial x}\ln(x^2 y)\vec{i} + \frac{\partial}{\partial y}\ln(x^2 y)\vec{j} = \frac{2}{x}\vec{i} + \frac{1}{y}\vec{j}$$

Example 4. Find the potential function for the field $\vec{F}(x,y) = (x-1)\vec{i} - (y+2)\vec{j}$.

For a vector field of the form $\vec{F}(x,y) = M\vec{i} + N\vec{j}$, M is the partial derivative with respect to x of the alleged potential function, and N is the partial derivative with respect to y of the alleged potential function. Integrate with respect to the appropriate variable to start with.

$$\int M\,dx = \int x - 1dx = \frac{1}{2}x^2 - x + g(y)$$

$$\int N dy = \int -y - 2dy = -\frac{1}{2}y^2 - 2y + h(x)$$

Any terms in either integral that have both x and y terms, must match or there is no potential function. In this case, there are no cross terms. The M integral can reconstruct the x-only terms of the potential function, but not the y-only terms. The N integral can reconstruct the y-only terms of the potential function. We need to combine the matching xy-terms, and the x-only terms from the first integral, and the y-only terms from the second integral to get the potential function. Here, our result is:

$$f(x,y) = \frac{1}{2}x^2 - x - \frac{1}{2}y^2 - 2y + K$$

Where K is some constant of integration.

Let's try a three-dimensional example.

Example 5. Find the potential function for the field $\vec{F}(x,y,z) = (yz - 2x + y)\vec{i} + (xz + 4y + x + 2z)\vec{j} + (xy + 2y - 3z^2)\vec{k}$.

I've picked a deliberately complicated one to see the kinds of things that can happen. As with the two-dimensional case, for a vector field of the form $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$, we need to integrate each term with respect to the appropriate variable, M with respect to x, N with respect to y, and now P with respect to z.

$$\int M\,dx = \int yz - 2x + ydx = xyz - x^2 + xy + g(y,z)$$

$$\int N dy = \int xz + 4y + x + 2zdy = xyz + 2y^2 + xy + 2yz + h(x,z)$$

$$\int P dz = \int xy + 2y - 3z^2 dz = xyz + 2yz - z^3 + j(x,y)$$

Any terms that have all three variables in them must match.

$$\int M\,dx = \int yz - 2x + y\,dx = xyz - x^2 + xy + g(y,z)$$
$$\int N\,dy = \int xz + 4y + x + 2z\,dy = xyz + 2y^2 + xy + 2yz + h(x,z)$$
$$\int P\,dz = \int xy + 2y - 3z^2\,dz = xyz + 2yz - z^3 + j(x,y)$$

The next check is that the xy terms in the M and N integrals must match, the yz terms in the N and P integrals must match, and the zx terms in the M and P integrals must match.

$$\int M\,dx = \int yz - 2x + y\,dx = xyz - x^2 + xy + g(y,z)$$
$$\int N\,dy = \int xz + 4y + x + 2z\,dy = xyz + 2y^2 + xy + 2yz + h(x,z)$$
$$\int P\,dz = \int xy + 2y - 3z^2\,dz = xyz + 2yz - z^3 + j(x,y)$$

There are no zx terms, and the others do match. All these go into the potential function. Also, any terms that have only one variable can only be reconstructed by the one integral, so they all go into the potential function.

$$f(x,y,z) = xyz + xy + 2yz - x^2 + 2y^2 - z^3 + K$$

Finding potential functions can be a hassle if there actually isn't one. Rather than try to calculate it, you can first test to see if it exists. In two dimensions, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then a potential function does exist. (This is like saying that $f_{xy} = f_{yx}$ for the unknown potential function.) In three dimensions, calculate the curl of the field. If the curl is identically zero, then the potential function does exist.

Practice Problems. Calculate the potential function, if it exists, or prove that it does not.

g. $\vec{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$
h. $\vec{F}(x,y) = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$
i. $\vec{F}(x,y,z) = (2xyz + z^2)\vec{i} + (x^2z - 3)\vec{j} + (x^2y + 2xz)\vec{k}$
j. $\vec{F}(x,y) = z\cos y\vec{i} - xz\sin y\vec{j} + x\cos y\vec{k}$