Betsy McCall

Integration by Parts

Integration by parts is used when two functions are multiplied together, but which are not related to each other through a chain rule (when substitution is used). Integration by parts is essentially in inverse operation on the product rule when taking derivatives, and that's where we get the formula $\int udv = uv - \int vdu$.

Integration by parts works best when the function we choose as *u* gets simpler when we take its derivative, and/or the function we choose as *dv* gets simpler when we integrate it. To ensure that, there is a basic rule of thumb we can use when choosing u (or we can choose dv using the reverse rule).

Choice	u	dv
1st	Logs: $\ln(x), \log_{10}(x)$	Exponential functions: $e^x, 2^x$, etc.
2nd	Algebraic functions (positive integer powers)	Algebraic function (negative integer powers)
3rd	Algebraic function (other)	Algebraic function (other)
4th	Exponential functions: $e^x, 2^x$, etc.	*

* These functions can't be integrated by themselves.

Not everything can be integrated by parts. Case in point: $\int \ln(x)e^x dx$. You can do one step of this problem, but the sequence of by-parts steps will never end.

Let's see some examples.

1. Logs and inverse trig functions by themselves can be integrated by parts.

   $\int \ln(x)dx$

   Choose $u = \ln x, dv = dx$. Find $du = \frac{1}{x}dx, v = x$

   $$\int \ln(x)dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$$

2. $\int x\ln(x+1)dx$

   Choose $u = \ln(x+1), dv = xdx$. Find $du = \frac{1}{x+1}dx, v = \frac{1}{2}x^2$

   $\int x\ln(x+1)dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int \frac{x^2}{x+1}dx$. This new integral requires long division.

   $$\frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\int x - 1 + \frac{1}{x+1}dx = \frac{1}{2}x^2\ln(x+1) - \frac{1}{2}\left[\frac{1}{2}x^2 - x + \ln(x+1)\right] + C$$

   $$= \frac{1}{2}\left[x^2\ln(x+1) + \frac{1}{2}x^2 - x + \ln(x+1)\right] + C$$

Betsy McCall

3. $\int x\sqrt{x+1}dx$

   This integral can be done by substitution as well (with $u=\sqrt{x+1}$), but we will do it by parts.

   Choose $u=x, dv=\left(x+1\right)^{1/2}dx$. Find $du=dx, v=\frac{2}{3}(x+1)^{3/2}$

   $$\int x\sqrt{x+1}dx=\frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\int (x+1)^{3/2}dx=\frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\left[\frac{2}{5}(x+1)^{5/2}\right]+C$$

4. $\int \frac{x}{(x+3)^2}dx$

   Choose $u=x, dv=(x+3)^{-2}dx$. Find $du=dx, v=-(x+3)^{-1}=-\frac{1}{x+3}$

   $$\int \frac{x}{(x+3)^2}dx=-\frac{x}{x+3}+\int \frac{1}{x+3}dx=-\frac{x}{x+3}+\ln(x+3)+C$$

5. $\int x^3e^{x^2}dx$

   Choose $u=x^2, dv=xe^{x^2}dx$. We do this because we will need to integrate dv by substitution. The exponential function here will not integrate without the x. Find $du=2xdx, v=\frac{1}{2}e^{x^2}$.

   $$\int x^3e^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\int \frac{1}{2}2xe^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\int xe^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$$

For some integrals, we will have to use by parts more than once. Check your answer by differentiating using the product rule.

6. $\int x^2e^xdx$

   Choose $u=x^2, dv=e^xdx$. Find $du=2xdx, v=e^x$.

   $$\int x^2e^xdx=x^2e^x-2\int xe^xdx$$

   For the new integral, choose $u=x, dv=e^xdx$, find $du=dx, v=e^x$. & integrate again.

   $$\int x^2e^xdx=x^2e^x-2\int xe^xdx=x^2e^x-2\left[xe^x-\int e^xdx\right]$$

   Apply by parts until you have an integral that can be integrated by substitution or a basic rule.

   $$\int x^2e^xdx=x^2e^x-2xe^x+2e^x+C$$

For very long by parts problems, where we will have to do integration by parts repeatedly, there is the tabular method to generate the terms we need.

7. $\int x^5e^{2x}dx$

   Choose $u=x^5, dv=e^{2x}dx$. Then create a table. Differentiate in the u column until you can't any more. And then in the dv column, integrate. In the first column, keep track of the signs.

Sign	u	dv
+	x^5	e^{2x}
-	$5x^4$	$\frac{1}{2}e^{2x}$
+	$20x^3$	$\frac{1}{4}e^{2x}$
-	$60x^2$	$\frac{1}{8}e^{2x}$
+	$120x$	$\frac{1}{16}e^{2x}$
-	120	$\frac{1}{32}e^{2x}$
+	0	$\frac{1}{64}e^{2x}$

Combine terms following the arrows:

$$\int x^5e^{2x}dx = \frac{1}{2}x^5e^{2x} - \frac{5}{4}x^4e^{2x} + \frac{5}{2}x^3e^{2x} - \frac{15}{4}x^2e^{2x} + \frac{15}{4}xe^{2x} - \frac{15}{8}e^{2x} + C$$

8. Some problems defy the general principles, and can only be done by trial and error. Case in point: $\int \frac{xe^{2x}}{(2x+1)^2}dx$

   You can try choosing according to the LIATE rule, and you will get nowhere. (Try it.) It just gets ugly. But there is a choice that does work: $u = xe^{2x}, dv = \left(2x+1\right)^{-2}dx$. Our choose for u will have to be differentiated using the product rule. $du = 2xe^{2x} + e^{2x} = e^{2x}(2x+1), v = -\frac{1}{2}\cdot\frac{1}{2x+1}$. Putting these into our formula:

$$\int \frac{xe^{2x}}{(2x+1)^2}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\int \frac{2x+1}{2x+1}e^{2x}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\int e^{2x}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{4}e^{2x} + C$$

Problems.

a. Choose *u* and *dv*. No need to integrate.
   - i. $\int \ln(x)dx$
   - ii. $\int x\ln(x)dx$
   - iii. $\int x^3e^{x^2}dx$
   - iv. $\int \frac{x}{\sqrt{3x+4}}dx$
   - v. $\int x^4e^{4x}dx$

b. Integrate.

vi. $\int \frac{x^2}{\sqrt[3]{2x+1}} dx$

vii. $\int x^2 e^{-3x} dx$

viii. $\int x^2 \ln(x) dx$

ix. $\int x \ln^2 x dx$

x. $\int \frac{xe^x}{(x+1)^2} dx$

xi. $\int x^4 e^{-4x} dx$

xii. $\int (2x-1)^4 2^{5x} dx$

c. Give five examples of integrals that cannot (or should not) be done with by parts.