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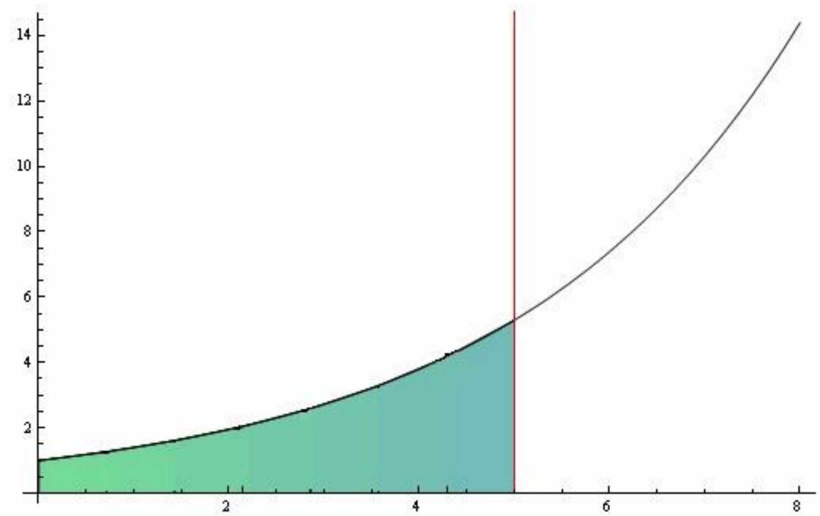
Definite Integrals
Estimating Area Under a Curve
(the long way)

Suppose that we could like to estimate the area under a the function $f(x) = e^{x/3}$, the area between the curve and the x-axis. Since the graph goes on forever, we have to specify which x-values we are interested in as well. Let's suppose between x=0 and x=5. The graph is shown below, and the shaded area is the area we wish to establish a value for.



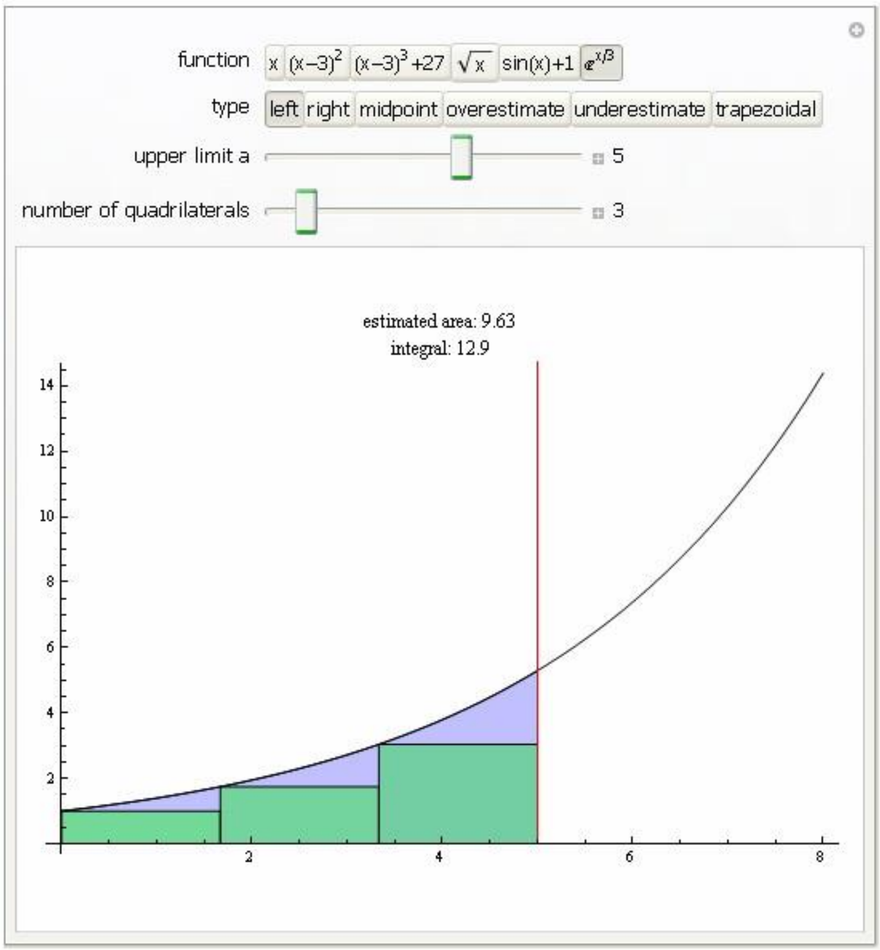
We don't really have a geometry formula that will work for such an odd shape, so how do we find it?

Well, we can start by estimating it. Suppose we divide the shape up into sections and use rectangles to estimate the area of each section? We might get a ballpark figure. Let's start with three sections. The total length in the x-direction is 5, so we'll let each section be 5/3 units wide. So we divide the sections at {0, 5/3, 10/3, 5}.

Since we are going to use rectangles to estimate the area in each section, we need heights for our rectangles. There are a number of ways we can make our estimates. We can choose an underestimate, or an overestimate, or a number of other methods that will get us a more accurate value (by adding complication). So let's start with something simple: let's start with an underestimate. We will get our estimate of the heights of our rectangles by using the height of the function at the lowest point on the interval. Since our function is increasing, we will choose the left side of each section. So on the interval [0,5/3], we will use the value of the function at

0, or f(0). The three heights we will need are then f(0), f(5/3) and f(10/3). We don't use f(5) because that is not the left end of one of our sections, and it would give us four rectangles and not three.

Let's see what this looks like:



We can get the estimate then by taking the height of each rectangle by the width of each rectangle and adding them up:

$$f(0)\frac{5}{3}+f\left(\frac{5}{3}\right)\frac{5}{3}+f\left(\frac{10}{3}\right)\frac{5}{3}=e^{0/3}\frac{5}{3}+e^{5/9}\frac{5}{3}+e^{10/3/3}\frac{5}{3}\approx 9.6344...$$

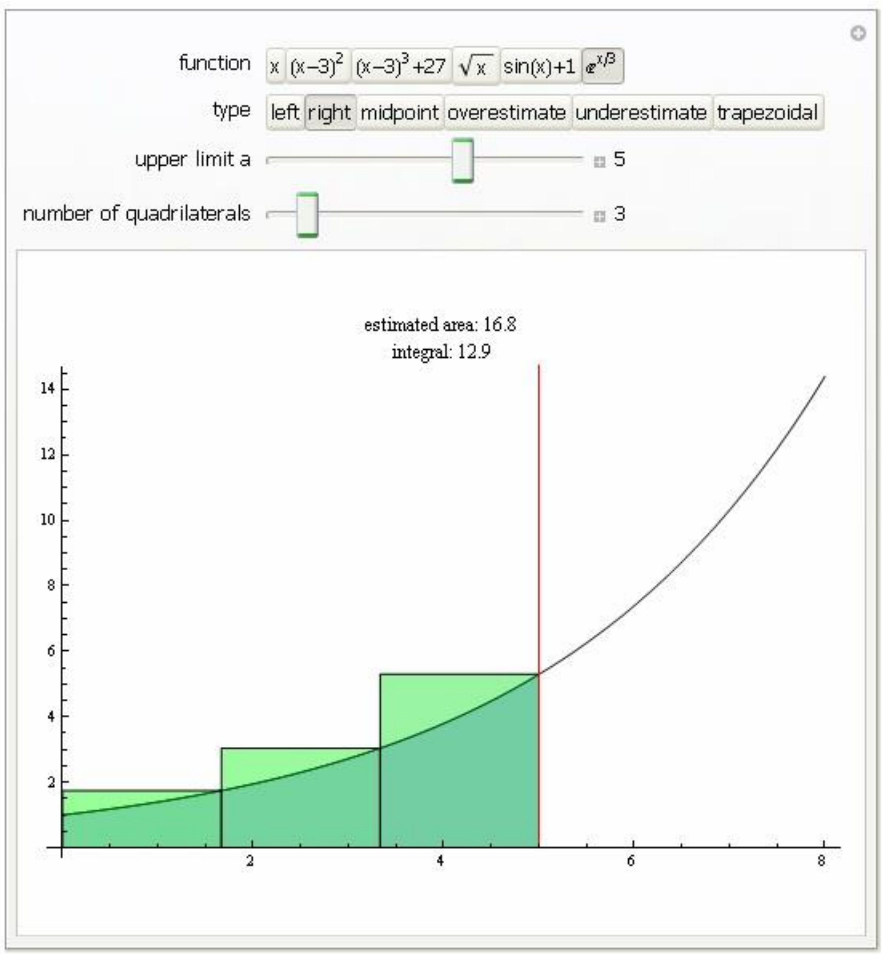
As you can see from the gaps between the rectangles and the curve, this isn't a very accurate estimate. We could try overestimating it, by using the height at the right side instead. While we do this, let's establish some notation that we'll need for the rest of the discussion. Since we have chosen to divide this area up into 3 sections, n=3. The starting x-value is sometimes called a, so a=0, or it can also be referred to as x_0. The ending value is called b, so b=5, or it can also be referred to as x_n. Why the difference? The values for a and b are given, while x_0 and x_n

come to play in our estimate. We found the length of each interval, which we will call Δx by taking the distance between *a* and *b* and dividing by the number of sections, i.e. $\Delta x = \frac{b-a}{n}$.
Each of the dividing points of our intervals becomes $x_i=\{a=x_0, x_1, x_2,\ldots, b=x_n\}$. In our example, we have $x_i=\{ a=x_0, x_1, x_2, b=x_3 \}=\{0, 5/3, 10/3, 5\}$.

In our overestimation, using the right side of the interval to make the rectangles larger, we'll be using the values 5/3, 10/3 and 5, and not using 0. Again, we have only three sections, and thus, three rectangles. Following the same pattern as before, the area of a rectangle is base*height gives:

$$f\left(\frac{5}{3}\right)\frac{5}{3}+f\left(\frac{10}{3}\right)\frac{5}{3}+f(5)\frac{5}{3}=e^{5/9}\frac{5}{3}+e^{10/9}\frac{5}{3}+e^{5/3}\frac{5}{3}\approx 16.79188\ldots$$

Here's what our estimate looks like:



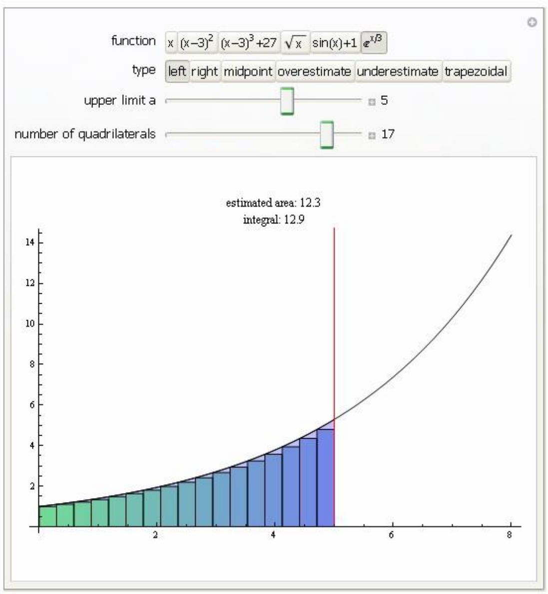
Using the notation we established this is the same as:

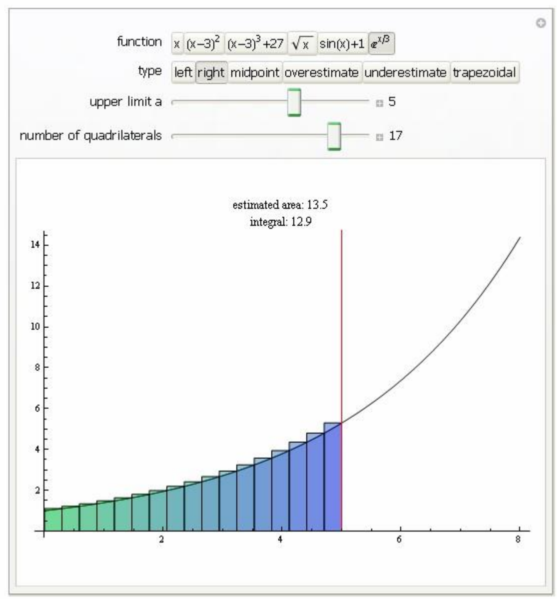
$$f(x_1)\frac{5}{3}+f(x_2)\frac{5}{3}+f(x_3)\frac{5}{3}=\left[f(x_1)+f(x_2)+f(x_3)\right]\frac{5}{3}=\left[f(x_1)+f(x_2)+f(x_3)\right]\Delta x$$

We can write this last expression in summation notation as:

$$\sum_{i=1}^{3} f(x_i)\Delta x$$

Well, this estimate is too big, clearly, but we can also see that the true value is somewhere in between these two numbers. How can we get a better estimate? Well, we can divide up the region into more subsections. Here's what the picture looks like for *n*=17.



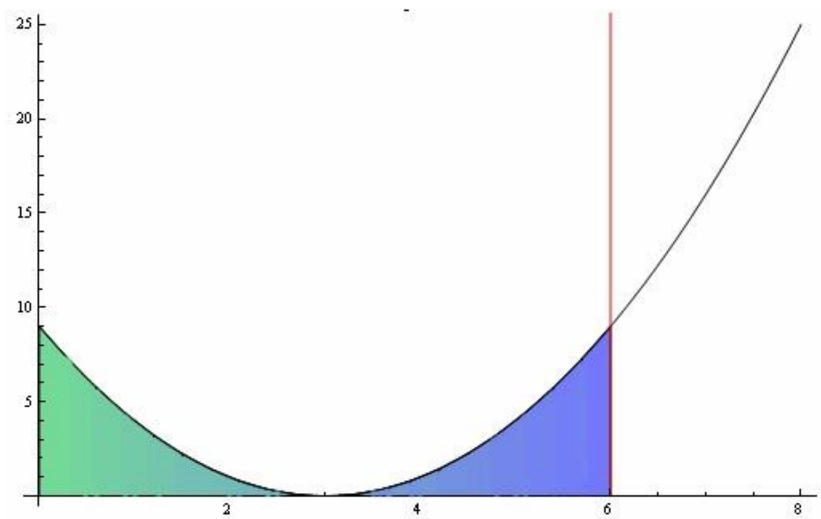


The first image uses the left side of the interval as our height estimate and underestimates the area. The second image uses the right side of the interval and overestimates the area. You can see from the images that the estimates in both cases are much closer, but still not close enough. But if we take the limit as *n* approaches infinity, it will equal the area we wish to measure. In other word, we want to solve the formula:

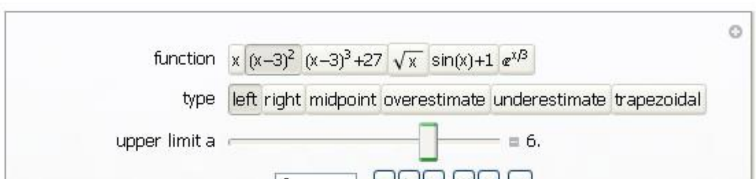
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = Area$$

How can we do that without calculating an infinite number of values?

Let's try this process again with a simpler function, like a polynomial.



The function shown above is $f(x)=(x-3)^2$ or if we expand it $f(x)=x^2-6x+9$. The shaded region is the area we wish to find, between x=0, and x=6. Because this function is not increasing or decreasing over the entire interval, choosing the left or right endpoints of our sections will not guarantee that we have an overestimate or an underestimate. Consider the rectangles obtained when n=3 and we use the left end of the interval to establish our rectangle height.



The first two are overestimates, but the third one is an underestimate. However, since we know that as n gets bigger, the estimate will get closer to the true value, it doesn't matter. We'd like to keep the math as simple as possible, so we will choose the left edge of the interval or the right edge of the interval and be consistent about it. The difference between the two formulas is shown below:

Left edge: $\sum_{i=1}^{n} f\left(x_{i-1}\right)\Delta x$ Right edge: $\sum_{i=1}^{n} f\left(x_i\right)\Delta x$

It turns out that that *i-1* subscript will add a little bit of algebra to our problem, so in the interest of simplicity, let's use the right-edge formula. In order to take the limit as *n* approached infinity, we will have to convert this summation into a formula involving *n*. Let's start with Δx. We will use the same $\Delta x = \frac{b-a}{n}$ that we use before, but we will be left with an expression containing *n*, since we won't be specifying it in advance. In this case, $\Delta x = \frac{6}{n}$.

Next, we to establish the x_i's. How did we do this before? We started with $a=x_0$, and then added Δx over and over again until we reached $b=x_n$. So here we have

$$\left\{0, 0+\frac{6}{n}, 0+\frac{6}{n}+\frac{6}{n}, 0+\frac{6}{n}+\frac{6}{n}+\frac{6}{n}, \ldots. 6-\frac{6}{n}, 6\right\} = \left\{0, 0+\frac{6i}{n}, \ldots, 6\right\} = x_i$$

where *i* is going to stand in for the number of times we've added *6/n* to get to that step. As our summation indicates, *i* goes up to *n*, and $\frac{6n}{n} = 6$, so we will stop at the correct place. Thus $x_i = a + i\Delta x = x_0 + i\Delta x$ in general.

$f(x_i)$ depends on the curve we are trying to find the area under. In this problem we will replace our values for x_i into the formula for the function.

$$f(x_i) = \left(\frac{6i}{n}\right)^2 - 6\left(\frac{6i}{n}\right) + 9 = \frac{36i^2}{n^2} - \frac{36i}{n} + 9$$

If we put this into the summation, together with our expression for Δx, we get:

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{36i^2}{n^2} - \frac{36i}{n} + 9\right]\left(\frac{6}{n}\right) = \sum_{i=1}^{n}\left[\frac{216i^2}{n^3} - \frac{216i}{n^2} + \frac{54}{n}\right]$$

In order to take the limit at this point, we still have *i*'s and summations in the equation; we will have to get rid of those in order to reduce everything to *n*'s. To do that, we need the following formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Let's simplify our expression so we can see how these go into our problem. We can use some basic algebra rules to separate our terms into separate sums, and factor anything out of each sum that is not an *i*.

$$\sum_{i=1}^{n}\left[\frac{216i^2}{n^3} - \frac{216i}{n^2} + \frac{54}{n}\right] = \sum_{i=1}^{n}\frac{216i^2}{n^3} - \sum_{i=1}^{n}\frac{216i}{n^2} + \sum_{i=1}^{n}\frac{54}{n} =$$

$$\frac{216}{n^3}\sum_{i=1}^{n} i^2 - \frac{216}{n^2}\sum_{i=1}^{n} i + \frac{54}{n}\sum_{i=1}^{n} 1$$

Using three of the summation formulas above, we can replace the summations in the expression.

$$\frac{216}{n^3}\sum_{i=1}^{n} i^2 - \frac{216}{n^2}\sum_{i=1}^{n} i + \frac{54}{n}\sum_{i=1}^{n} 1 = \frac{216}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{216}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{54}{n}[n]$$

Simplify the expression to something as simple as possible by expanding the parentheses and distributing, and combining like terms.

$$\frac{216}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{216}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{54}{n}[n] =$$

$$\frac{216}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] - \frac{216}{n^2}\left[\frac{n^2+n}{2}\right] + 54 =$$

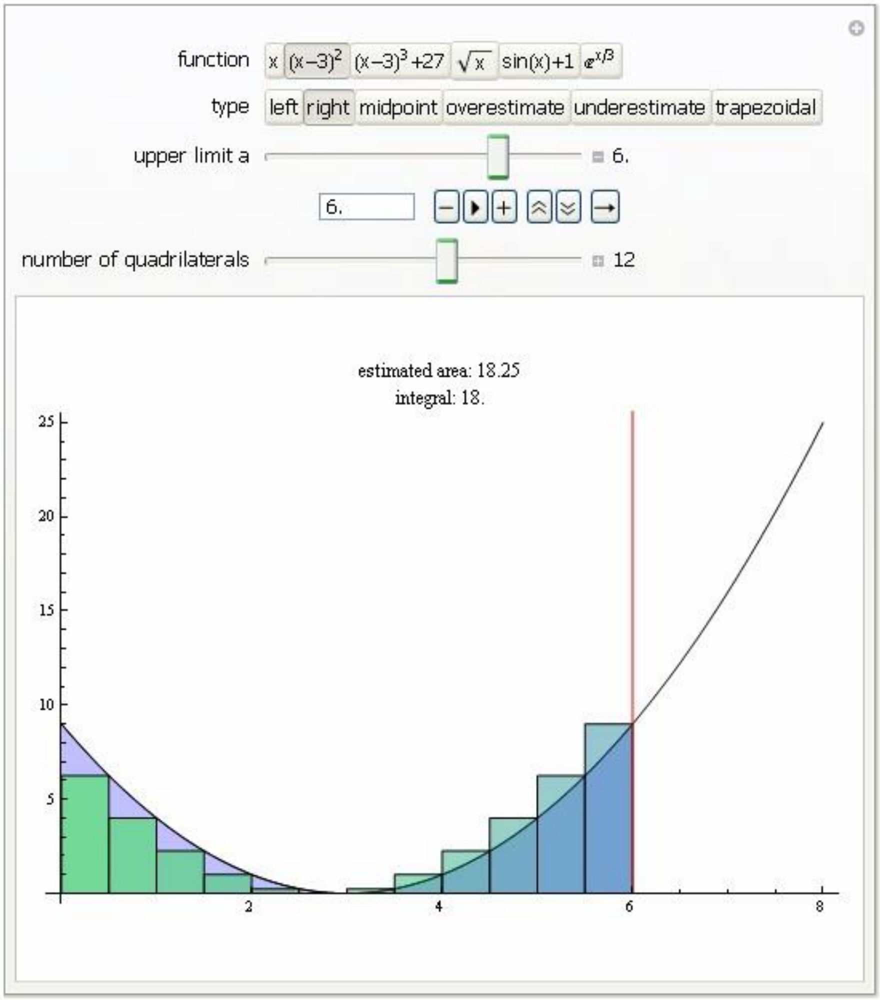
$$\frac{36}{n^3}\left[2n^3+3n^2+n\right] - \frac{108}{n^2}\left[n^2+n\right] + 54 =$$

$$\frac{72n^3}{n^3} + \frac{108n^2}{n^3} + \frac{36n}{n^3} - \frac{108n^2}{n^2} - \frac{108n}{n^2} + 54 =$$

$$72 + \frac{108}{n} + \frac{36}{n^2} - 108 - \frac{108}{n} + 54 =$$

$$18 + \frac{36}{n^2}$$

We can use this expression to find the area estimate under the curve for a particular value of *n*. Suppose that we let *n*=12.



$$18+\frac{36}{12^2}=18+\frac{36}{144}=18+\frac{1}{4}=18.25$$

We can also use this expression to find the limit as *n* approaches infinity.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x=\lim_{n\to\infty}\left(18+\frac{36}{n^2}\right)=18$$

The value of the area that we find is called the definite integral and can be notated as:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x=\int_a^b f(x)dx$$

We chose a very specific case for this example to make the algebra as simple as possible, but this idea can be further generalized into Riemann sums, where even the width of each section can vary, under the condition that as n gets bigger, all the sections get smaller.

Area Under a Curve (the short way!)

The Fundamental Theorem of Calculus allows us to short-cut the longer algebraic process (you still have to know the long process!). The Fundamental Theorem of Calculus says:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where *F(x)* is an anti-derivative of the function *f(x)*. To use our previous example with this method, we get:

$$\int_0^6 x^2 - 6x + 9dx = \frac{x^3}{3} - 3x^2 + 9x\Big|_0^6 = \left(\frac{6^3}{3} - 3\left(6^2\right) + 9\left(6\right)\right) - \left(0\right) = 72 - 108 + 54 = 18$$

Phew!!