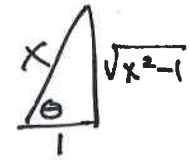
Trig Substitution

i. $\int \frac{1}{x\sqrt{x^2-1}}\,dx$ $x = \sec\theta$ $\sqrt{x^2-1} = \tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$\int \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{\cancel{\sec\theta}\cdot\cancel{\tan\theta}} = \int d\theta = \theta + C$



$x = \sec\theta \Rightarrow \sec^{-1}x = \theta$

$= \operatorname{arcsec} x = \theta$

$= \operatorname{arcsec} x + C$

which is as expected.

ii. $\int x^2\sqrt{1-x^2}\,dx$ $x = \sin\theta$ $dx = \cos\theta$

$x^2 = \sin^2\theta$ $\sqrt{1-x^2} = \cos\theta$

$\int \sin^2\theta\cos\theta\cos\theta\,d\theta = \int \sin^2\theta\cos^2\theta\,d\theta =$

$\int \frac{1}{2}(1-\cos 2\theta)\left(\frac{1}{2}\right)(1+\cos 2\theta)\,d\theta = \frac{1}{4}\int 1 - \cos^2 2\theta\,d\theta =$

$\frac{1}{4}\int 1 - \frac{1}{2}(1+\cos 4\theta)\,d\theta = \frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos 4\theta\,d\theta =$

$\frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4\theta\,d\theta = \frac{1}{8}\int 1 - \cos 4\theta\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right] + C$

to convert back to x we need all θ's not 2θ or 4θ.

$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin 4\theta = 2(2\sin\theta\cos\theta)(1-2\sin^2\theta)$

$= 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$

$\frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right] + C = \frac{1}{8}\left[\theta - \sin\theta\cos\theta + 2\sin^3\theta\cos\theta\right] + C$

$x = \sin\theta$ $\arcsin x = \theta$

$= \frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2} + \frac{1}{4}x^3\sqrt{1-x^2} + C$

iii. $\int x^2(1+x^2)^{3/2}\,dx$

$x=\tan\theta \qquad dx=\sec^2\theta\,d\theta$

$x^2=\tan^2\theta \qquad \left(\sqrt{1+x^2}\right)^3=\sec^3\theta$

$\int \tan^2\theta\sec^3\theta\sec^2\theta\,d\theta$

$\int \tan^2\theta(1+\tan^2\theta)\sec^2\theta\sec\theta\,d\theta$

$= \int \tan^2\theta\sec^2\theta\sec\theta\,d\theta \quad + \quad \int \tan^4\sec^2\theta\sec\theta\,d\theta$

$u=\sec\theta \qquad dv=\tan^2\theta\sec^2\theta\,d\theta \qquad\qquad u=\sec\theta \qquad dv=\tan^4\theta\sec^2\theta\,d\theta$

$du=\sec\theta\tan\theta\,d\theta \qquad v=\frac{1}{3}\tan^3\theta \qquad\qquad du=\sec\theta\tan\theta \qquad v=\frac{1}{5}\tan^5\theta$ *

$\frac{1}{3}\tan^3\theta\sec\theta-\int\frac{1}{3}\sec\theta\tan^4\theta\,d\theta$ 　　　　we'll come back to this *

$\frac{1}{3}\tan^3\theta\sec\theta-\frac{1}{3}\int\sec\theta(\sec^2\theta-1)\tan^2\theta\,d\theta$

$\frac{1}{3}\tan^3\theta\sec\theta-\frac{1}{3}\int\sec^3\theta\tan^2\theta\,d\theta+\frac{1}{3}\int\sec\theta\tan^2\theta\,d\theta$

$\int\tan^2\theta\sec^3\theta\,d\theta=\frac{1}{3}\tan^3\theta\sec\theta-\frac{1}{3}\int\sec^3\theta\tan^2\theta\,d\theta+\frac{1}{3}\int\sec\theta\tan^2\theta\,d\theta$

$+\frac{1}{3}\int\tan^2\theta\sec^3\theta\,d\theta \qquad\qquad +\frac{1}{3}\int\sec^3\theta\tan^2\theta\,d\theta$

$\frac{3}{4}\cdot\frac{4}{3}\int\tan^2\theta\sec^3\theta\,d\theta=\frac{3}{4}\cdot\frac{1}{3}\tan^3\theta\sec\theta+\frac{3}{4}\cdot\frac{1}{3}\int\sec\theta\tan^2\theta\,d\theta$

$\int\tan^2\theta\sec^3\theta\,d\theta=\frac{1}{4}\tan^3\theta\sec\theta+\frac{1}{4}\int\sec\theta\tan^2\theta\,d\theta$

$\int\sec\theta\tan^2\theta\,d\theta = \int\sec\theta(\sec^2\theta-1)\,d\theta=\int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta$

$\int\sec^3\theta\,d\theta = \qquad u=\sec\theta \qquad dv=\sec^2\theta$

$du=\sec\theta\tan\theta\,d\theta \qquad v=\tan\theta$

$\sec\theta\tan\theta-\int\sec\theta\tan^2\theta\,d\theta$

$\int\sec\theta\tan^2\theta\,d\theta=\sec\theta\tan\theta-\int\sec\theta\tan^2\theta\,d\theta-\ln|\sec\theta+\tan\theta|+C$

$+\int\sec\theta\tan^2\theta\,d\theta \qquad\qquad +\int\sec\theta\tan^2\theta\,d\theta$

$\cancel{2}\int\sec\theta\tan^2\theta\,d\theta=\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln|\sec\theta+\tan\theta|+C$

Thus $\int\tan^2\theta\sec^3\theta\,d\theta=\frac{1}{4}\tan^3\theta\sec\theta+\frac{1}{8}\sec\theta\tan\theta-\frac{1}{8}\ln|\sec\theta+\tan\theta|+C$

Still need to do $\int\tan^4\sec^2\theta\,d\theta$ but will reduce to what we've done

$$\int \tan^4\theta \sec^2\theta \sec\theta \, d\theta = \int \tan^4\theta \sec^3\theta \, d\theta$$

$$u = \sec\theta \qquad dv = \tan^4\theta \sec^2\theta \, d\theta$$

$$du = \sec\theta \tan\theta \, d\theta \qquad v = \tfrac{1}{5}\tan^5\theta$$

$$\tfrac{1}{5}\tan^5\theta \sec\theta - \tfrac{1}{5}\int \sec\theta \tan^6\theta \, d\theta$$

$$\tfrac{1}{5}\tan^5\theta \sec\theta - \tfrac{1}{5}\left[\int \sec\theta (\sec^2 - 1)\tan^4\theta \, d\theta =\right.$$

$$\tfrac{1}{5}\tan^5\theta \sec\theta + \tfrac{1}{5}\int \sec\theta \tan^4\theta \, d\theta - \tfrac{1}{5}\int \sec^3\theta \tan^4\theta \, d\theta = \int \tan^4\theta \sec^3\theta \, d\theta$$

$$+\tfrac{1}{5}\int \sec^3\theta \tan^4\theta \, d\theta + \tfrac{1}{5}\int \tan^4\theta \sec^3\theta \, d\theta$$

$$\tfrac{\cancel{5}}{6}\tfrac{1}{5}\tan^5\theta \sec\theta + \tfrac{\cancel{5}}{6}\tfrac{1}{5}\int \sec\theta \tan^4\theta \, d\theta = \tfrac{\cancel{5}}{\cancel{6}}\tfrac{\cancel{6}}{5}\int \tan^4\theta \sec^3\theta \, d\theta$$

$$\tfrac{1}{6}\tan^5\theta \sec\theta + \tfrac{1}{6}\int \sec\theta \tan^4\theta \, d\theta = \int \tan^4\theta \sec^3\theta \, d\theta$$

$$\int \sec\theta (\sec^2 - 1)\tan^2\theta \, d\theta =$$

$$\int \tan^2\theta \sec^3\theta \, d\theta - \int \tan^2\theta \sec\theta \, d\theta =$$

$$\tfrac{1}{4}\tan^3\theta \sec\theta + \tfrac{1}{8}\sec\theta \tan\theta - \tfrac{1}{8}\ln|\sec\theta + \tan\theta| + \cancel{C} - \tfrac{1}{2}\sec\theta \tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| \; {*C}$$

$$\tfrac{1}{4}\tan^3\theta \sec\theta - \tfrac{3}{8}\sec\theta \tan\theta + \tfrac{3}{8}\ln|\sec\theta + \tan\theta| + C$$

$$\int \tan^4 \sec^3\theta \, d\theta =$$

$$\tfrac{1}{6}\tan^5\theta \sec\theta + \tfrac{1}{6}\left[\tfrac{1}{4}\tan^3\theta \sec\theta - \tfrac{3}{8}\tan\theta \sec\theta + \tfrac{3}{8}\ln|\sec\theta + \tan\theta|\right] + C$$

$$\tfrac{1}{6}\tan^5\theta \sec\theta + \tfrac{1}{24}\tan^3\theta \sec\theta - \tfrac{1}{16}\tan\theta \sec\theta + \tfrac{1}{16}\ln|\sec\theta + \tan\theta| + C$$

$$\int \tan^2 \sec^5\theta \, d\theta = \int \tan^2\theta \sec^3\theta \, d\theta + \int \tan^4 \sec^3\theta \, d\theta =$$

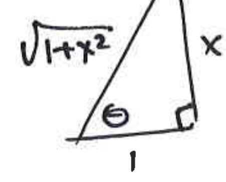
$$\tfrac{1}{4}\tan^3\theta \sec\theta + \tfrac{1}{8}\sec\theta \tan\theta - \tfrac{1}{8}\ln|\sec\theta + \tan\theta| +$$

$$\tfrac{1}{6}\tan^5\theta \sec\theta + \tfrac{1}{24}\tan^3\theta \sec\theta - \tfrac{1}{16}\tan\theta \sec\theta + \tfrac{1}{16}\ln|\sec\theta + \tan\theta|$$

$$\tfrac{1}{6}\tan^5\theta \sec\theta + \tfrac{7}{24}\tan^3\theta \sec\theta + \tfrac{1}{16}\sec\theta \tan\theta - \tfrac{1}{16}\ln|\sec\theta + \tan\theta| + C$$

$$\frac{x}{1} = \tan\theta$$

$$\sec\theta = \frac{\sqrt{1+x^2}}{1}$$



$\frac{1}{6}x^5\sqrt{1+x^2} + \frac{7}{24}x^3\sqrt{1+x^2} + \frac{1}{16}x\sqrt{1+x^2} - \frac{1}{16}\ln\left|\sqrt{1+x^2}+x\right| + C$

iv. $\int \frac{x}{\sqrt{2x-x^2}}\,dx = \int \frac{x}{\sqrt{1-(1-2x+x^2)}}\,dx = \int \frac{x}{\sqrt{1-(x-1)^2}}\,dx$

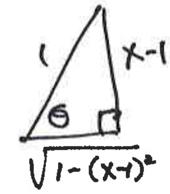
$x-1 = \sin\theta \qquad \sqrt{1-(x-1)^2} = \cos\theta$

$dx = \cos\theta \qquad x = \sin\theta + 1$

$\int \frac{\sin\theta+1}{\cancel{\cos\theta}}\,\cancel{\cos\theta}\,d\theta = \int \sin\theta + 1\,d\theta =$

$-\cos\theta + \theta + C$

$-\sqrt{2x-x^2} + \arcsin(x-1) + C$



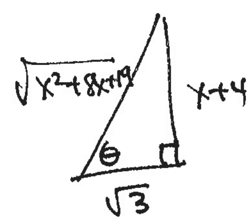
v. $\int \frac{1}{\sqrt{x^2+8x+19}}\,dx = \int \frac{1}{\sqrt{(x^2+8x+16)+3}}\,dx = \int \frac{1}{\sqrt{(x+4)^2+3}}\,dx$

$x+4 = \sqrt{3}\tan\theta \qquad \sqrt{(x+4)^2+3} = \sqrt{3}\sec\theta.$

$dx = \sqrt{3}\sec^2\theta\,d\theta$

$\frac{x+4}{\sqrt{3}} = \tan\theta$

$\int \frac{\cancel{\sqrt{3}}\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sqrt{3}\sec\theta}} = \int \sec\theta\,d\theta$



$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{x^2+8x+19}}{\sqrt{3}} + \frac{x+4}{\sqrt{3}}\right| + C$

Sometimes you will see the √3 factored out & included w/ the +C

$\ln\left|\frac{\sqrt{x^2+8x+19}+x+4}{\sqrt{3}}\right| + C = \ln\left|\sqrt{x^2+8x+19}+x+4\right| - \ln|\sqrt{3}| + C$

constant

$= \ln\left|\sqrt{x^2+8x+19}+x+4\right| + C$

vi. $\int \frac{x^2}{\sqrt{x^2-6x-5}}\,dx = \int \frac{x^2}{\sqrt{(x^2-6x+9)-5-9}}\,dx = \int \frac{x^2}{\sqrt{(x-3)^2-14}}\,dx$

$x-3 = \sqrt{14}\sec\theta$ $\qquad \sqrt{(x-3)^2-14} = \sqrt{14}\tan\theta$

$dx = \sqrt{14}\sec\theta\tan\theta\, d\theta$

$x^2 = (\sqrt{14}\sec\theta + 3)^2 =$

$14\sec^2\theta + 6\sqrt{14}\sec\theta + 9$

$$\int \frac{(14\sec^2\theta + 6\sqrt{14}\sec\theta + 9)\cdot \sec\theta\cancel{\tan\theta}\cancel{\sqrt{14}}\,d\theta}{\cancel{\sqrt{14}}\cancel{\tan\theta}} =$$

$$\int 14\sec^3\theta + 6\sqrt{14}\sec^2\theta + 9\sec\theta\, d\theta$$

$\int \sec^3\theta\, d\theta$ $\qquad u = \sec\theta$ $\qquad dv = \sec^2\theta$

$\qquad du = \sec\theta\tan\theta\, d\theta$ $\qquad v = \tan\theta\, d\theta$

$\int \sec^3 d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$

$\int \sec^3 d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$

$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\, d\theta$

$+\int \sec^3\theta\, d\theta \qquad\qquad + \int \sec^3\theta\, d\theta$

$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$

$\int \sec^2\theta\, d\theta = \tan\theta$

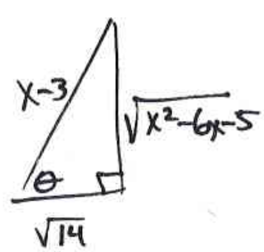
$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$

$\int 14\sec^3\theta + 6\sqrt{14}\sec^2\theta + 9\sec\theta\, d\theta =$

$7\sec\theta\tan\theta + 7\ln|\sec\theta + \tan\theta| + 6\sqrt{14}\tan\theta + 9\ln|\sec\theta + \tan\theta| + C$

$7\sec\theta\tan\theta + 16\ln|\sec\theta + \tan\theta| + 6\sqrt{14}\tan\theta + C$

$7\left(\frac{x-3}{\sqrt{14}}\right)\frac{\sqrt{x^2-6x-5}}{\sqrt{14}} + 16\ln\left|\frac{x-3}{\sqrt{14}} + \frac{\sqrt{x^2-6x-5}}{\sqrt{14}}\right| + 6\cancel{\sqrt{14}}\frac{\sqrt{x^2-6x-5}}{\cancel{\sqrt{14}}} + C$ $\qquad \frac{x-3}{\sqrt{14}} = \sec\theta$



$\frac{x-3}{2}(\sqrt{x^2-6x-5}) + 16\ln\left|\frac{x-3+\sqrt{x^2-6x-5}}{\sqrt{14}}\right| + 6\sqrt{x^2-6x-5} + C$

vii. $\int \frac{x^5}{\sqrt{8-x^2}}\,dx$ $\qquad x=\sqrt{8}\sin\theta \qquad \sqrt{8-x^2}=\sqrt{8}\cos\theta$

$dx=\sqrt{8}\cos\theta\, d\theta \qquad x^5=(\sqrt{8}\sin\theta)^5=64\sqrt{8}\sin^5\theta$

$$\int \frac{64\sqrt{8}\sin^5\theta\, \cancel{\cos\theta}\, d\theta\, \cancel{\sqrt{8}}}{\cancel{\sqrt{8}}\, \cancel{\cos\theta}} = 64\sqrt{8}\int \sin^5\theta\, d\theta =$$

$$64\sqrt{8}\int \sin\theta\,(1-\cos^2\theta)^2\, d\theta$$

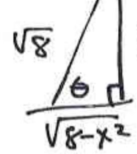
$u=\cos\theta \quad -du=+\sin\theta\, d\theta$

$$-64\sqrt{8}\int (1-u^2)^2\, du =$$

$$-64\sqrt{8}\int 1-2u^2+u^4\, du = -64\sqrt{8}\left[u-\frac{2}{3}u^3+\frac{1}{5}u^5\right]+C$$

$$= -\underline{64\sqrt{8}}\left[\cos\theta-\frac{2}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta\right]+C$$

$$-64\sqrt{8}\left[\frac{\sqrt{8-x^2}}{\sqrt{8}}-\frac{2}{3}\left(\frac{\sqrt{8-x^2}}{\sqrt{8}}\right)^3+\frac{1}{5}\left(\frac{\sqrt{8-x^2}}{\sqrt{8}}\right)^5\right]+C \qquad \frac{x}{\sqrt{8}}=\sin\theta$$



$$-64\sqrt{8-x^2}+\frac{16}{3}\left(\sqrt{8-x^2}\right)^3-\frac{1}{5}\left(\sqrt{8-x^2}\right)^5+C$$

viii. $\int \frac{3e^{4x}}{(9-e^{2x})^{3/2}}\,dx$ $\qquad u=e^x$

$du=e^x dx$

$$\int \frac{3e^{3x}\cdot e^x\,dx}{(9-(e^x)^2)^{3/2}} = \int \frac{3u^3\,du}{(9-u^2)^{3/2}}$$

$u=3\sin\theta$

$du=3\cos\theta$

$(9-u^2)^{3/2}=(3\cos\theta)^3=$

$27\cos^3\theta$

$u^3=27\sin^3\theta$

$$\int \frac{3\cdot\cancel{27}\sin^3\theta\cdot 3\cancel{\cos\theta}}{\cancel{27}\cos^{\cancel{3}}_{2}\theta}\,d\theta =$$

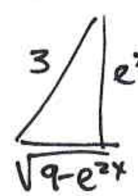
$$9\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = 9\int \frac{(1-\cos^2\theta)\sin\theta}{\cos^2\theta}\,d\theta = 9\int \frac{1}{\cos^2\theta}\sin\theta - \sin\theta\, d\theta$$

$\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} \qquad \frac{u}{3}=\frac{e^x}{3}=\sin\theta$

$$=9\int \sec\theta\tan\theta - \sin\theta\, d\theta =$$

$$9\sec\theta+9\cos\theta+C$$

$$9\cdot\frac{3}{\sqrt{9-e^{2x}}}+\frac{9\cdot\sqrt{9-e^{2x}}}{3}+C=\frac{27}{\sqrt{9-e^{2x}}}+3\sqrt{9-e^{2x}}+C$$



ix. $\int \frac{x^7}{\sqrt{1+x^4}}\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du \; x\,dx$

$\int \frac{(x^2)^3\,x\,dx}{\sqrt{1+(x^2)^2}} =$

$u = \tan\theta \qquad u^3 = \tan^3\theta$

$du = \sec^2\theta\,d\theta$

$\sqrt{1+u^2} = \sec\theta$

$\frac{1}{2}\int \frac{u^3\,du}{\sqrt{1+u^2}} =$

$\frac{1}{2}\int \frac{\tan^3\theta\,\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec\theta}} = \frac{1}{2}\int \tan^3\theta\,\sec\theta\,d\theta =$

$\frac{1}{2}\int (\sec^2\theta - 1)\tan\theta\,\sec\theta\,d\theta =$

$w = \sec\theta$

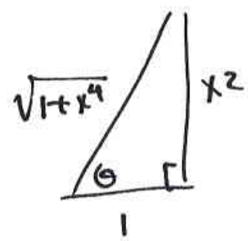
$dw = \sec\theta\tan\theta\,d\theta$

$\frac{1}{2}\int w^2 - 1\,dw =$

$u = \frac{x^2}{1} = \tan\theta$

$\frac{1}{2}\left[\frac{1}{3}w^3 - w\right] + C = \frac{1}{6}w^3 - \frac{1}{2}w + C =$

$\frac{1}{6}\sec^3\theta - \frac{1}{2}\sec\theta + C$



$\frac{1}{6}\left(\sqrt{1+x^4}\right)^3 - \frac{1}{2}\sqrt{1+x^4} + C$