Work

In your introductory physics classes, work is given by the formula W=FD [W=work, F=force, D=distance]. But this only works in very limits circumstances where both the force and the distance are fixed (all parts of a solid object are being moved the same distance in a fixed force field; for instance, lifting a book off the table, since the book is not being moved far enough that the gravity changes significantly from starting to stopping point). But what if we fly into space? Now, the gravitational force is changing significantly (not to mention the lost rocket fuel), so this old method with grossly overestimate the work done. Or, if we pull on a spring, the further we pull the spring, the stronger the force gets. If you empty a tank of water (or wind up a chain), the water from the top is not being moved as far as the water being pumped out of the bottom of the tank. All of these problems will require us to integrate to solve.

The spring examples are the simplest, so let's start there.

Example 1. The force on a spring is given by $F = kx$ where k is a constant that depends on the spring and x is the distance from its natural length. This is Hooke's Law. For most of these problems you will need to find k from information provided, and then calculate the real problem.

Q. Suppose it takes 16 pounds of force to stretch a spring 6 inches. How much work is done stretching the spring 6 more inches?

One important thing to remember is the need to use feet in English units. So first convert 6 inches to ½ foot.

A. If $F = kx$, then $16 = \frac{1}{2}k$. Therefore, for this spring $F = 32x$, where x is in inches, and F is in pounds. If we want to pull it six more inches, then we are pulling it from 6 inches to 12 inches or ½ foot to 1 foot. Our work integral is like W=FD, but since the force is changing we use $W = \int_{start}^{stop} F(x)dx$ where the dx acts like the distance. So here:

$$W = \int_{\frac{1}{2}}^{1}(32x)dx = 16x^2|_{\frac{1}{2}}^{1} = 16(1)^2 - 16\left(\frac{1}{2}\right)^2 = 16 - 4 = 12 \text{ foot-pounds.}$$

If you integrate work in inches, then you willget inch-pounds, instead, 144 inch-pounds. It is better use feet since foot-pounds is a more standard unit, and if you are given mass rather than weight and need to convert, you will use gravity, which is measured, in English units, in feet/s^2. In metric, use meters (unless you are doing astrophysics, for some reason).

Practice Problems:

1. Use Hooke's Law to find the work done stretching a spring from its rest length to 4 inches.
2. Use Hooke's Law to find the work done compressing a spring from 3 inches less than its rest length to 7 inches less than its rest length.

Do problems 1 & 2 for the following springs:

a. 70 lbs. of force is required to stretch the spring 10 inches.
b. 30 lbs. of force is required to compress the spring 1 foot.

You may neglect the negative sing that may change from stretching to compressing. The magnitude will be the same. Do be careful with units.

Propulsions problems are similar to Hooke's Law if we neglect things like air resistance and the expenditure of fuel to get a rocket into orbit. Our force law is just a bit different: $F = G\frac{m_1 m_2}{d^2}$ where G is the gravitational constant, m_1 and m_2 are the masses of the objects involved (in most problems, one of these objects is the Earth), and d is the distance between the object. However, if we know a condition of the system, say, the weight (force) of the object on the surface, then we can combine three of these constants into one unknown: $F = \frac{k}{x^2}$. In this case, x is not the distance we are moving from the surface, but the distance from the center of gravity = the center of the Earth. You can assume for the purposes of these problems that the radius of the Earth is 4000 miles, or about 6400 km. This is the distance from the center to the surface and our starting point.

Example 2. Suppose we want to send a rocket into orbit 1000 miles above the Earth. The rocket weighs 5 tons at launch. Calculate the work done getting it into orbit, neglecting air resistance and spent fuel.

On Earth: $F = \frac{k}{x^2} \Rightarrow 5 = \frac{k}{4000^2} \Rightarrow k = 8\times 10^7 \Rightarrow F = \frac{8\times 10^7}{x^2}$

$$W = \int_{4000}^{4000+1000} \frac{8\times 10^7}{x^2}\,dx = 8\times 10^7 \int_{4000}^{5000} x^{-2}\,dx = 8\times 10^7 \left[-\frac{1}{x}\right]_{4000}^{5000} = 8\times 10^7 \left[-\frac{1}{5000} + \frac{1}{4000}\right] = \frac{8\times 10^7}{20000} =$$

$4000\ mile-tons$

This is much less than if we calculated W=FD because, as we noted earlier, the force is getting weaker as we go away from the planet. In a physics course, you can do this problem more precisely by accounting for air resistance, or by writing the equation to account for changing fuel weight as it is burned during launch. Both are changing during flight and will eventually vanish. But that is beyond what we'd like to worry about for this course.

Practice Problems.

3. Suppose that you are launching an 11 ton satellite into orbit around Earth. How much work is done getting the ship into an orbit a) 500 miles high? b) Into the same orbit as the Moon? The Moon's orbit is 385,000 km from the Earth's center and 11 tons is approximately 475,000 newtons. (Neglect the moon's gravity.)
4. Suppose you are launching a satellite from the surface of the moon back to Earth? How much work is needed? (Neglect the Earth's gravity.) The radius of the moon is approximately 1700 km (1100 miles), and the surface gravity is approximately 1/6th that of Earth's.
5. A more realistic model includes both the gravity of the Earth and the gravity of the Moon. Since the forces are pulling in opposite directions, the work will have opposite signs. Combine the results of the previous two calculations to determine the work done. At what distance does the work done by each body cancel? (Set the upper limit to be a dummy variable, and then solve the resulting algebraic equation.)

A similar example can be done with electrical force, since the force only depends on the distance between the two particles (if we assume the specific charges are fixed).

Example 3. Suppose there are two electrical charges one is at the origin (0,0) and one at (3,4). The force on the second particle is currently feeling a repulsive force of 16 newtons. Calculate the work needed to move the particle from (3,4) to the point (1,2).

The distance between the two particles starts at 5 units (think of them as meters for force in newtons). You get the distance from the distance formula.

$r_{start} = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5$

The force is calculated from a simplified version of Coulomb's Law $F = \frac{k}{r^2}$, where r is the distance. I'm using r here instead of x so as not to confuse it with the x-coordinate of the points. Use the 5 and 16 to calculate the constant k. $-16 = \frac{k}{5^2} \Rightarrow k = -400$. The force is negative since it's repulsive rather than attractive (compare with gravity).

The new distance is at (1,2) so $r_{stop} = \sqrt{(1-0)^2 + (2-0)^2}\sqrt{1+4} = \sqrt{5}$. So now integrate:

$$W = \int_5^{\sqrt{5}} F(r)dr = \int_5^{\sqrt{5}} \frac{-400}{r^2} dr = \left.\frac{400}{r}\right|_5^{\sqrt{5}} = \frac{400}{\sqrt{5}} - \frac{400}{5} = \frac{80(\sqrt{5}-1)}{5} \approx 98.995 \text{ newton-meters.}$$

It is possible for the work to be negative, but in this example, we would need to be going in the direction of the force (i.e. away from the origin).

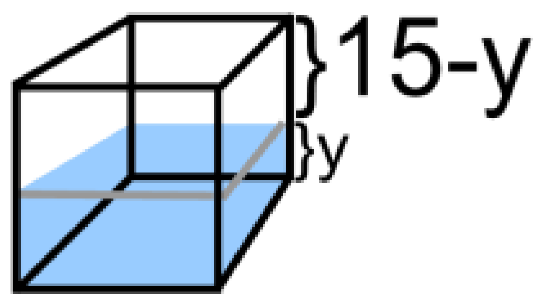
Practice Problems.

6. Move the same particles as in the example from (3,4) to (7,8).
7. To calculate the energy in a particle configuration, one just needs to calculate the work needed to move the particle from infinity (where the force is zero) to its current position. Use the force data in the example to calculate this.

The second type of problem is a bit more complicated because more of it needs to be derived from the geometry of the individual problem. As before, we'll be needing two things: a force expression and a distance travelled expression.

Example 4. Consider a rectangular tank with a base of 5 feet by 6 feet and a height of 15 feet. How much work is required to empty half the tank if the tank is full? The density of water is 62.4 lbs./feet³ (or 9800 newtons/meter³).

To calculate the force component we need to calculate $F = volume \times density$. The volume of a slice of the water in the tank is the area of the base 5x6 times an infinitesimal slice of height dy. We want an infinitesimal slice because each slice is moved a different distance and so requires a changing amount of work. $F = 30 \times 62.4dy$



For a slice at height y, the distance the water has to be moved is 15 feet, minus the distance it started at (15-y). Finally, if we are emptying half the tank (the top half), our work integral becomes:

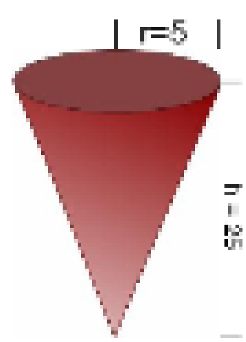
$$W = \int_{15/2}^{15} 30 \times 62.4(15-y)dy = 1872\int_{15/2}^{15} 15 - ydy = 1872\left[15y - \frac{1}{2}y^2\right]_{15/2}^{15} =$$

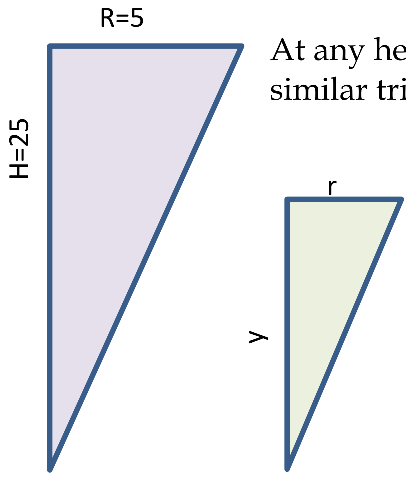
$$1872\left[225 - \frac{225}{2} - \frac{225}{2} + \frac{225}{8}\right] = 52,650 \ \textit{foot} - \textit{pounds}$$

Is the work the same if you pump out the bottom of the tank?

So the issue for these problems comes down to the geometry of the tanks.

- What is your tank is cylindrical? The base area of a cylinder is a circle ($A = \pi r^2$). For a cylinder, r is constant and the problem will tell you what it is, but be careful of units.
- What if your tank is conical? Suppose your conical tank is 10 feet across the top and 25 feet tall? You need an equation for the radius (5 along the top) at any place in the cone, since we will need it for our cylindrical slice to approximate the volume.





At any height in the tank, the filled portion of the tank forms a similar triangle with the triangular profile of the entire tank.

From properties of similar triangles, we know that the ratios of sides remains the same: $\frac{radius}{height} = \frac{5}{25} = \frac{r}{y}$.

We can solve for the radius formula we need: $r = \frac{y}{5}$, and this goes into the area formula for the cylinder.

The volume then is $V = \pi r^2 dy = \pi \left(\frac{y}{5}\right)^2 dy$.

- ❖ What if you have a spherical or hemispherical tank? Again, our slices will be cylindrical, so what we need is a formula for the radius in terms of the height. Recall that the equation for a circle is $x^2 + y^2 = r^2$ where r is the radius of the sphere and x is the radius we are looking for. Suppose our tank has a radius of 12 feet, then our equation is $x^2 + y^2 = 144 \Rightarrow x = \sqrt{144 - y^2}$. So the volume would be $V = \pi R^2 dy = \pi \left(\sqrt{144 - y^2}\right)^2 dy = \pi (144 - y^2) dy$. This will be true if the origin is at ground level. If the tank is moved off the ground, your circle equation will need to be $x^2 + (y-k)^2 = r^2$, where k is how high the center of the circle is moved off the ground. Consider a water tower is 25 feet off the ground, the center is 25+12=37 feet off the ground: $x^2 + (y-37)^2 = 144 \Rightarrow x = \sqrt{144 - (y-37)^2}$. The square root will cancel in the volume formula, but there will be a bit of algebra to do before you can integrate.

Practice Problems.

8. Consider a rectangular tank whose base is 15 feet by 12 feet and with a height of 20 feet. It is also raised off the ground by 35 feet. How much work does it take to fill the tank?
9. Consider a cylindrical well with a diameter of 9 inches and a depth of 140 feet. How much work is needed to drain the water to a depth of 25 feet.
10. Consider a conical tank with diameter 14 feet and a height of 12 feet. Find the amount of work needed to drain the tank if it is only half full.
11. Consider a spherical tank described above: of radius 12 feet whose base is 25 feet off the ground. Find the work needed to fill the tank.
12. Suppose we have a hemispherical tank buried 10 feet underground with a radius of 16 feet. Find the work needed to empty the tank.

Problems involving chains are quite similar, but now the force will be given by $F = density \times length$, and the length will just be dy.

Example 5. Suppose a chain of length 200 feet is hanging from a crane. The chain weighs 14 lbs. per foot. Calculate the work done in winding up half the chain (i.e. to 100 feet).

$F = density \times length = 14dy$. For a segment of chain at height y, it has to be wound (200-y), so our integral is: $W = \int_0^{100} 14(200 - y)dy$. Here, we can go from the bottom since it is the bottom of the chain that is being moved, a bit like the top is not moving at all.

$$14\int_0^{100} (200 - y)dy = 14\left[200y - \frac{1}{2}y^2\right]_0^{100} = 14\left[20000 - \frac{10000}{2}\right] = 210{,}000 \text{ foot-pounds.}$$

If we added a dead weight to the end of the chain, like a wrecking ball that doesn't change, you can calculate that work as just W=FD, where F is just the weight of the wrecking ball.

Practice Problems.

13. Suppose you have a 150 foot chain that weighs 2 lbs. per foot. Calculate the work needed to wind up the entire chain.
14. If a 50 lbs. hook is added to the bottom of the chain, what is the total work in winding up the chain? (Just add on the work of the hook to #13.)
15. What is the work done in taking the same chain in #13, but pulling the bottom up to the top so that the chain hangs doubled.