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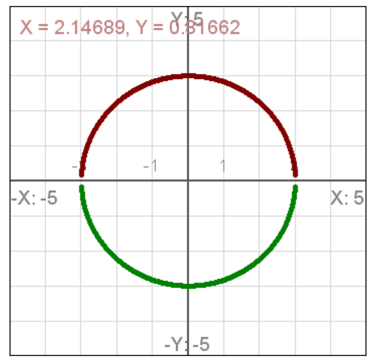
## 3-D Surfaces

3-D surfaces can be difficult to visualize without some practice. You will need to be able to visualize in 3-D in order to succeed at multivariable calculus. To improve your visualization skills, it's a good idea to find a program (online or computer software) that will allow you to graph and manipulate 3-D graphs to get a better idea of what they look like. We will cover some basic graph types from various angles here, but there is really no substitute for rotating the graphs yourself. One piece of software you can download for free is GraphCalc, from graphcalc.com.

Let's start with solids of rotation. We worked with these back when we did the Shell and Disk/Washer method without developing the equations for the surfaces.

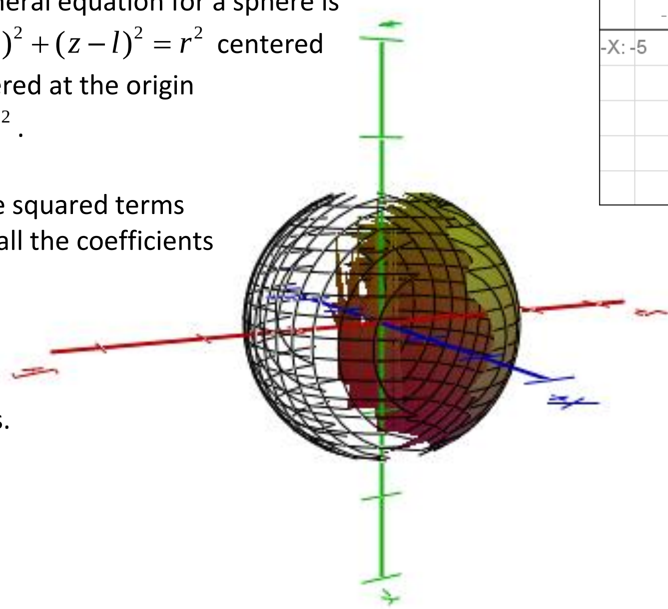
1. **Sphere**.

   If in two-dimensions we have a circle given by the pair of functions $f(x)=\sqrt{r^2-x^2}$ and $g(x)=-\sqrt{r^2-x^2}$, our 2-D circle looks like this. If we revolve this graph around either the x- or y-axis, we get a sphere. Here the two halves of the sphere are shaded differently to enhance contrast. The general equation for a sphere is $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ centered at (h,k,l) or centered at the origin $x^2+y^2+z^2=r^2$.



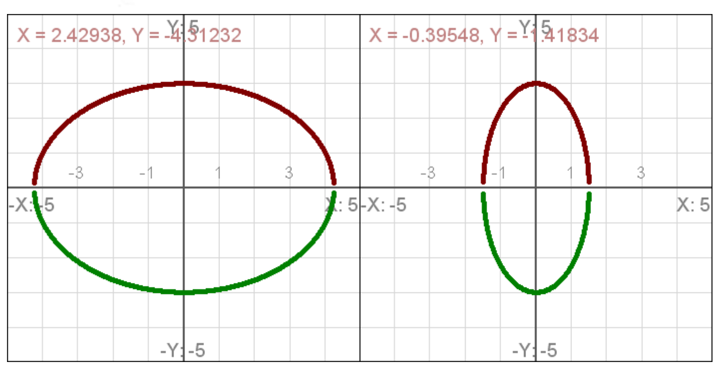

   Notice that all the squared terms are positive, and all the coefficients are the same.

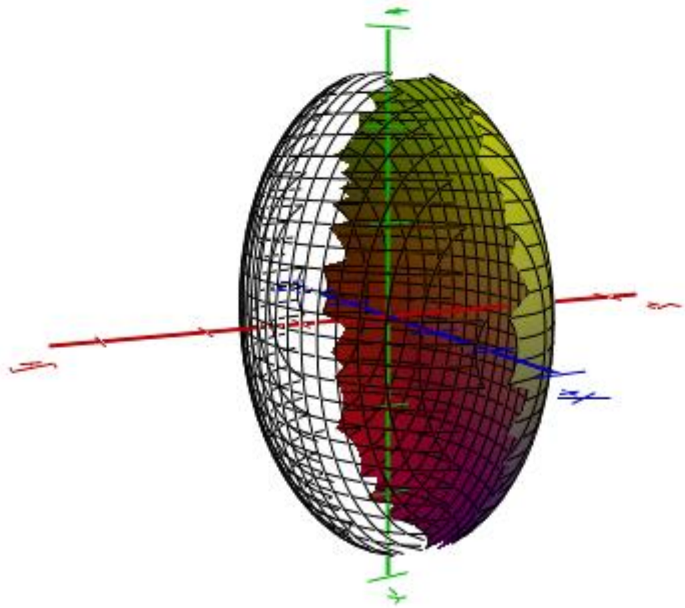
   This graph will look like a circle in all three planes.

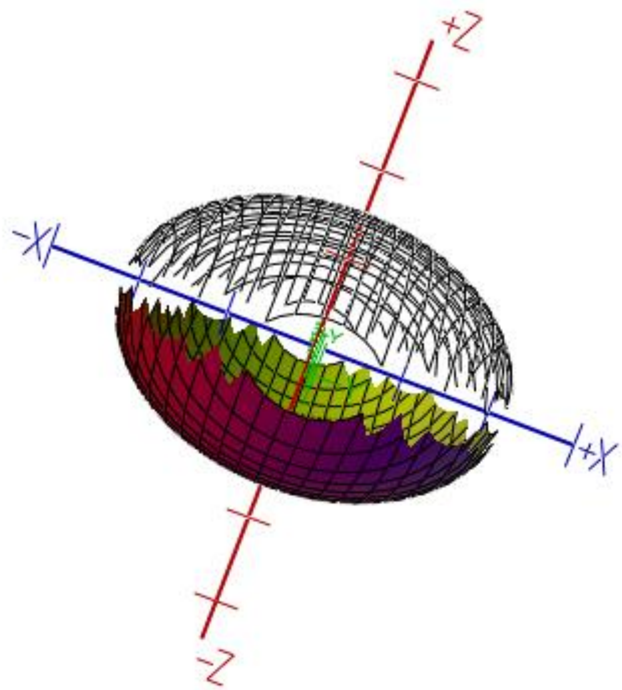


2. **Ellipsoid**.

   If we start instead with an ellipse, the long direction can either be in x or y, and we revolve it around either axis, we get an ellipsoid, and a particular kind of ellipsoid, where two axes of the ellipse are the same. The general ellipsoid looks quite similar, but it can have three different axis lengths.



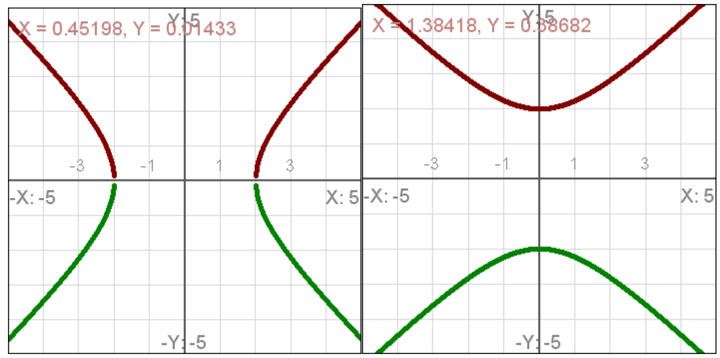




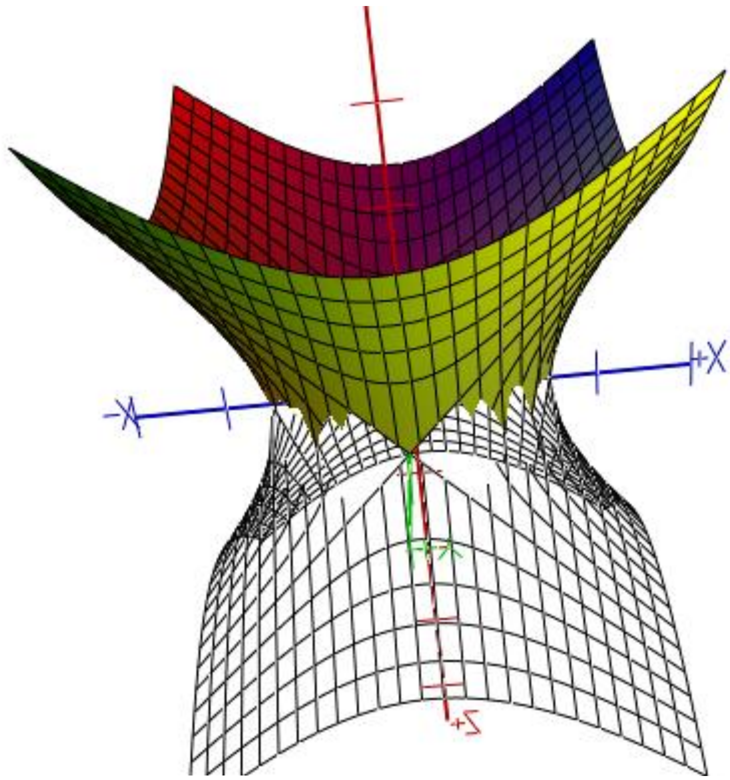

The equation of an ellipsoid looks very similar to the sphere equation, but the coefficients are not the same.

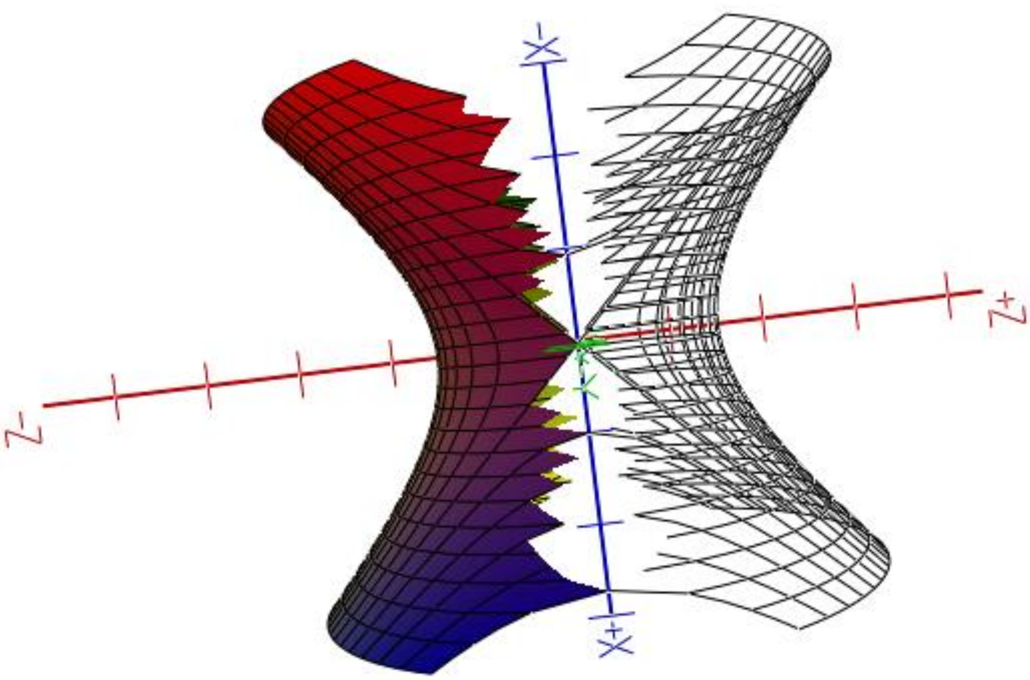
Centered at the origin, we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. It is standard for $c \le b \le a$, i.e. for a to define the longest axis. The general ellipsoid looks like an ellipse in every plane, but if b=c or b=a, then one plane will look like a circle.

3. **Hyperboloid of One Sheet**. Hyperboloids of one sheet are generated from hyperbolas rotated around the axis perpendicular to the transverse axis, so that as the graph is rotated, the two halves of the hyperbola merge. In other words, rotate the graph on the left around the y-axis, or rotate the graph on the right around the x-axis.





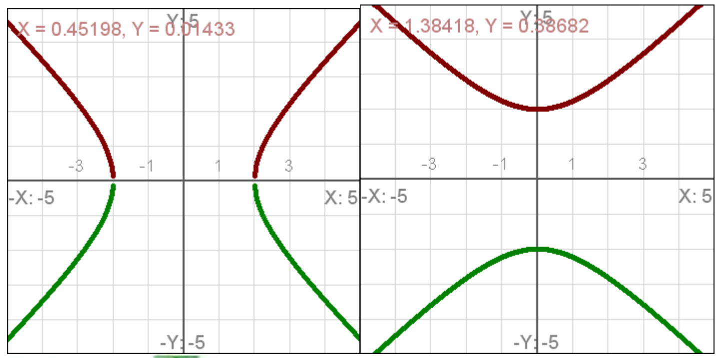



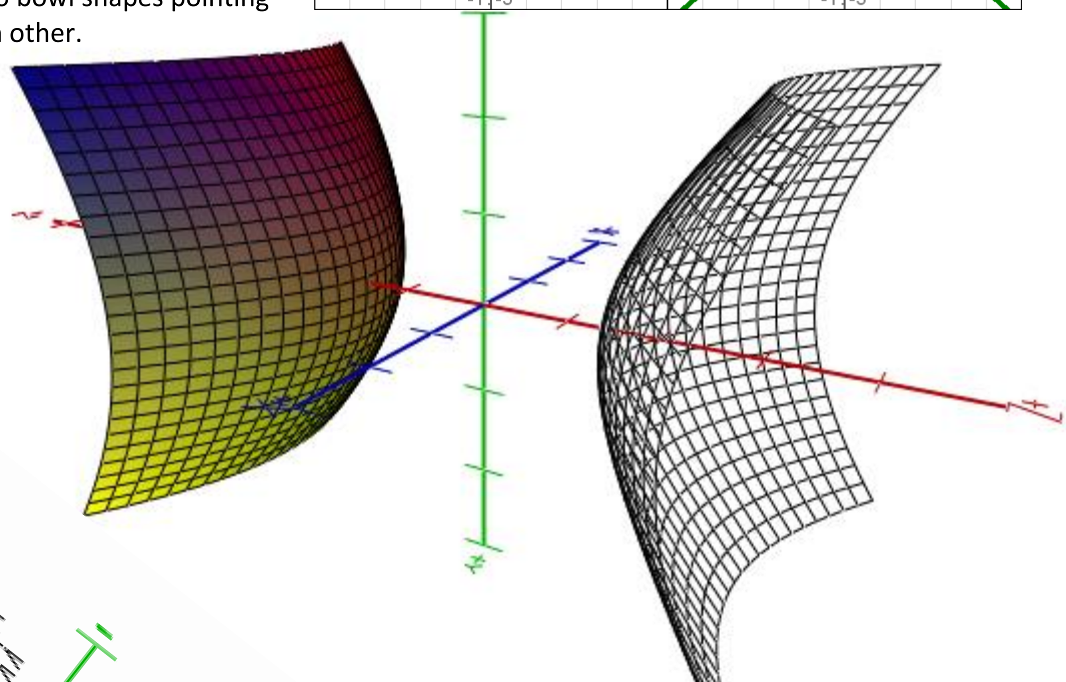
For the hyperboloid of one sheet, the graph looks like a hyperbola on two planes, and either an ellipse or a circle on the other. The graph is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, or any variation of this where just one of the terms is negative. The variable which is negative defines the direction of the axis of rotation (the one going through the middle of the sheet).

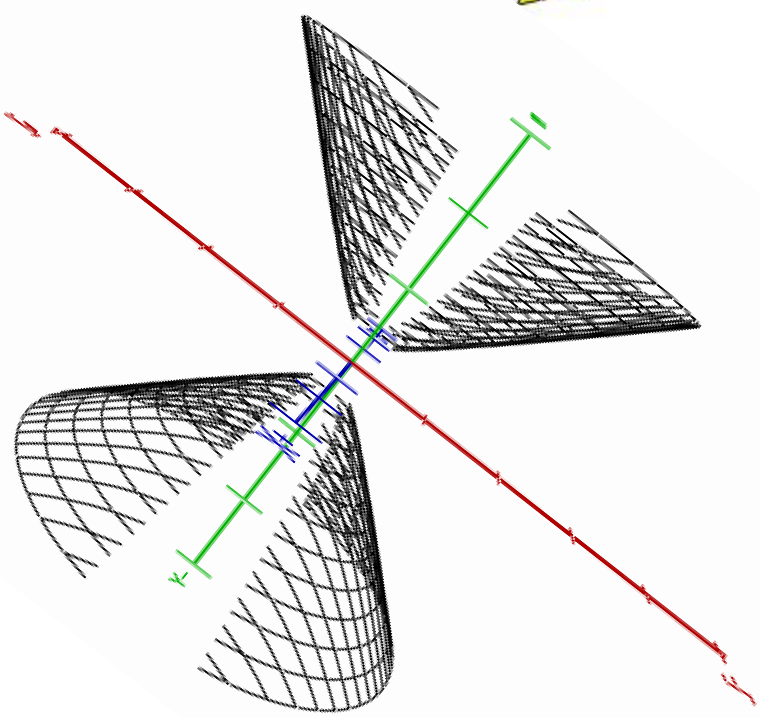
4. **Hyperboloid of Two Sheets**.

For the hyperboloid of two sheets , we start with the same hyperbola graph as before, but now we rotate the graphs around the transverse axis (the one that goes through the middle of the graph). For the left graph that's the x-axis; for the right graph that's the y-axis.



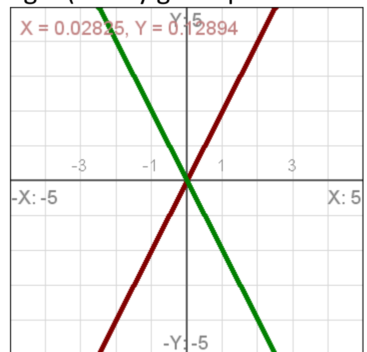

The result is two bowl shapes pointing away from each other.





The equation for the hyperboloid of two sheets is the same as the one for one sheet except for there being two negative signs: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. Like with the hyperboloid of one sheet, two planes give an ellipse, but the third plane has no intersection. Any plane parallel to that which also intersects the surface will have the trace of an ellipse or a circle. The transverse axis is the one that is positive.

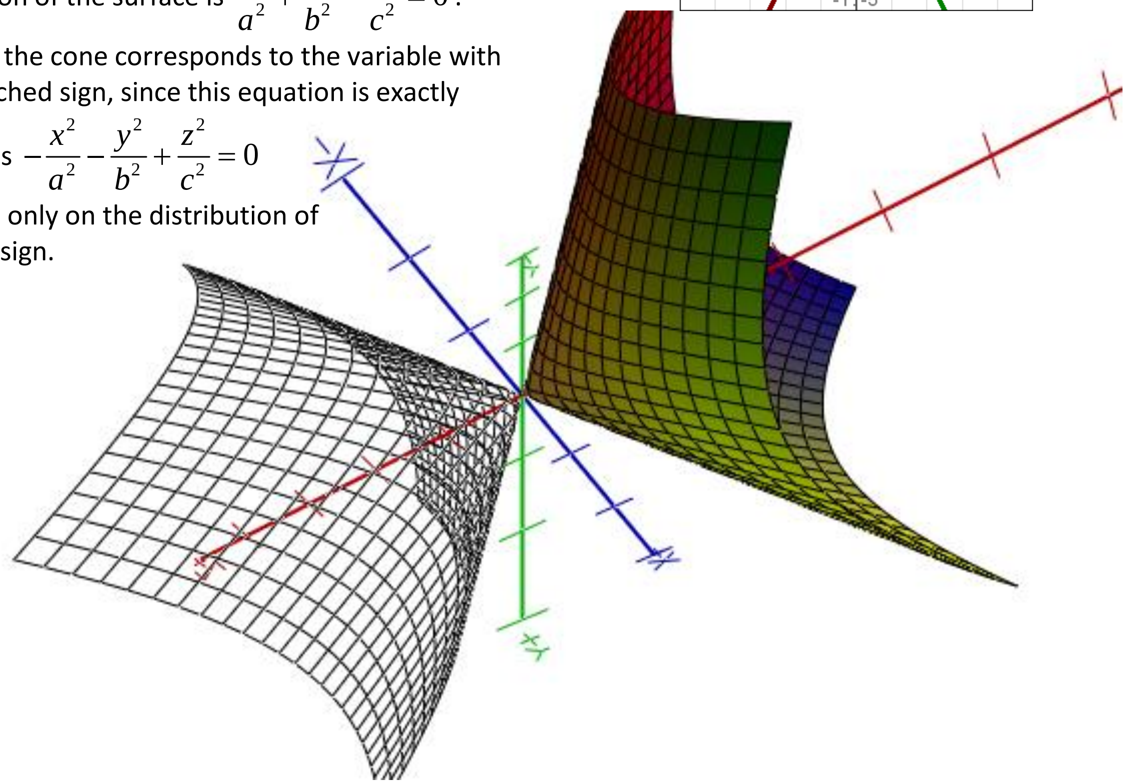
5. **Elliptic (or Circular) Cone**.
By continually reducing the central radius of the hyperboloid of one sheet, or the gap between the two sections of the hyperboloid of one sheet, to zero, you create an elliptic cone. As a surface of revolution, this generated by a line passing through the origin (or any given point (h,k,l) and rotated around one axis (passing through the center).



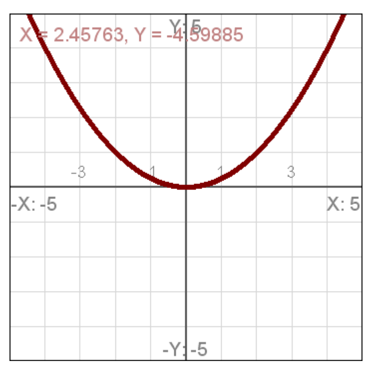

Passing through two planes, the graph will look like a x-shape, but planes parallel to those will have the shape of a hyperbola. The remaining plane passing through the center will have just a point, but planes parallel to that will have an elliptical (or circular) trace.

The equation of the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$.

The axis of the cone corresponds to the variable with the unmatched sign, since this equation is exactly the same as $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$ dependent only on the distribution of a negative sign.

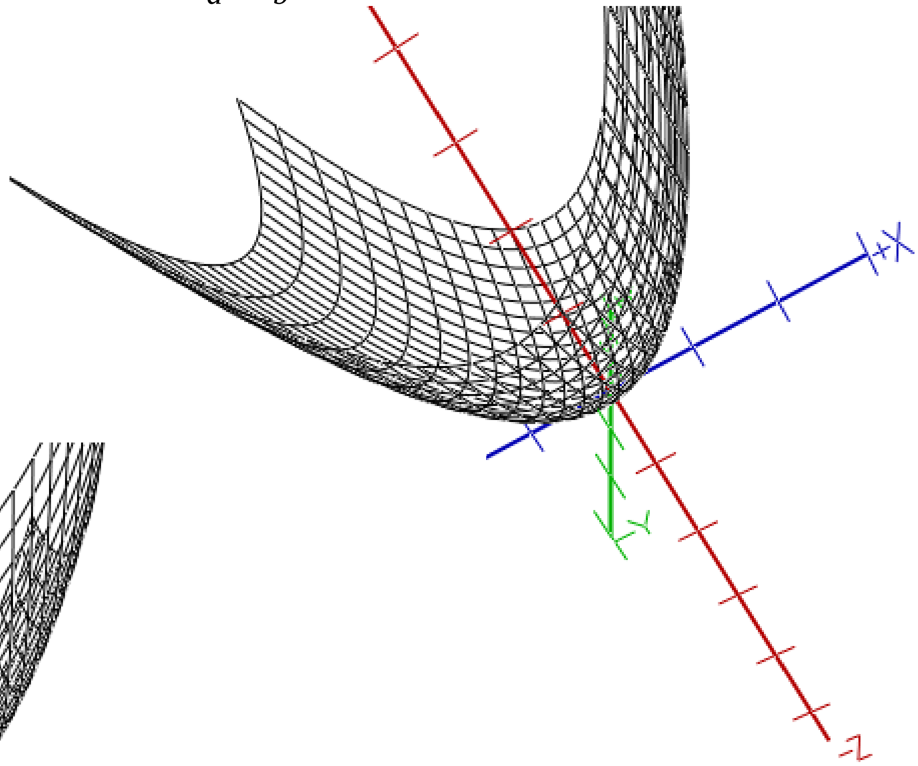


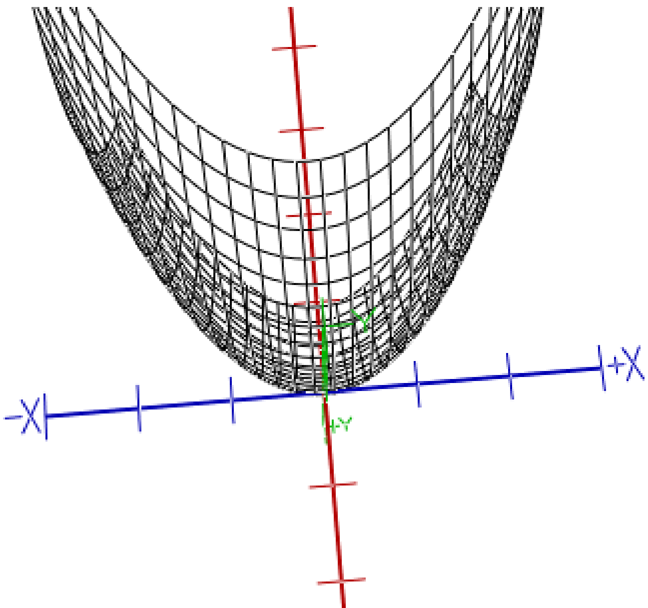
6. **Elliptic Paraboloid**.
Of all the graphs we've considered so far, the paraboloid is our first function in three dimensions. Starting with the graph of a parabola, we rotate this graph around the y-axis (or if the graph is another orientation, its axis of symmetry).




In two planes this graph will appear as a parabola, but in the third plane, it will appear as an ellipse (or circle), unless it touches only the vertex, in which case it will be just a point.

The axis of the parabola is determined by the linear variable in the equation $z = x^2 + y^2$ (circular trace) or $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (elliptical trace).

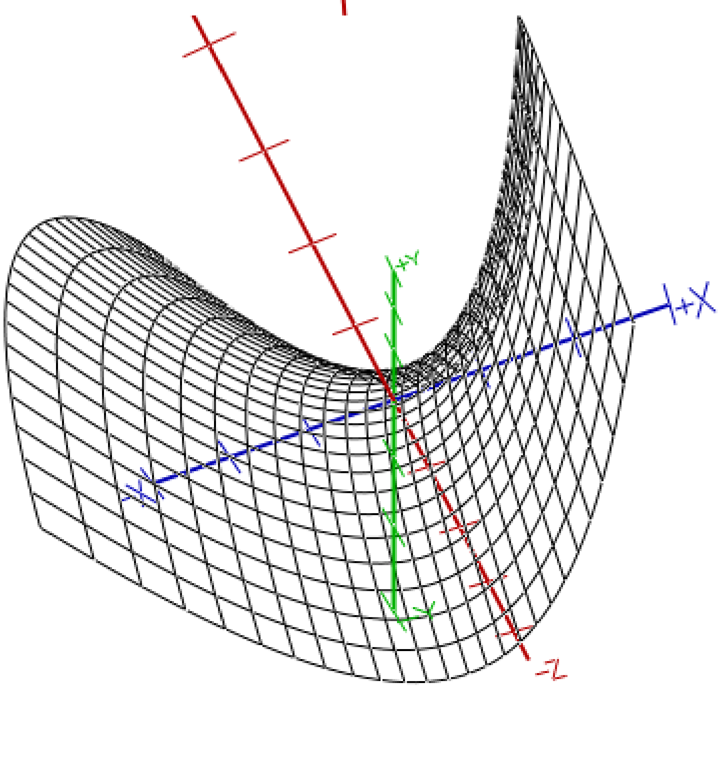


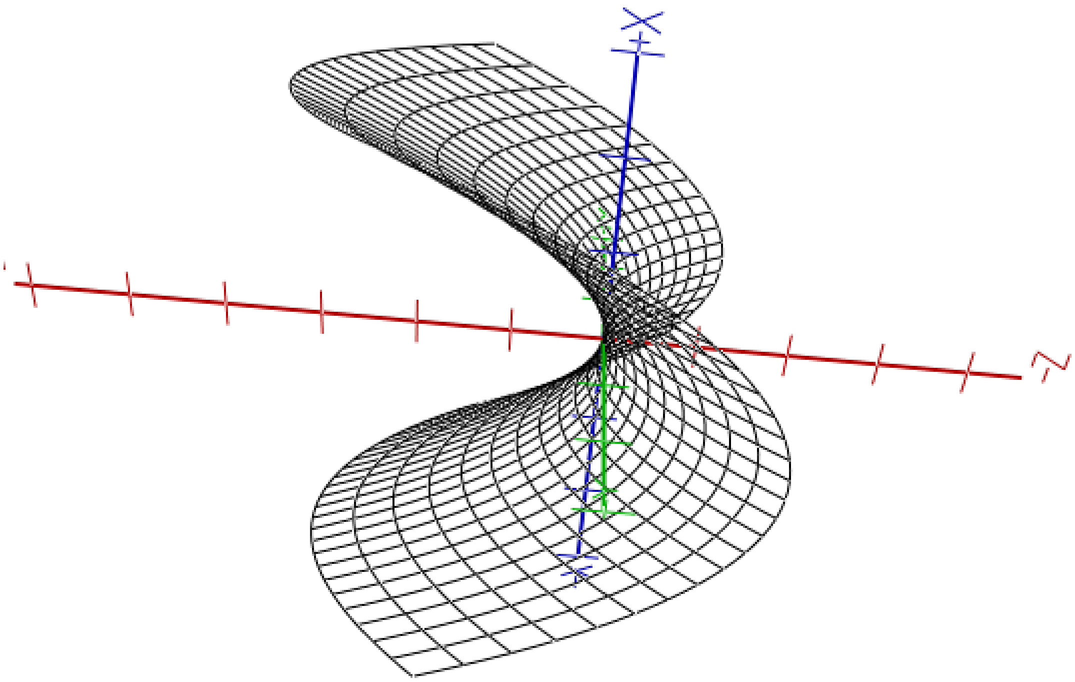




7. **Hyperbolic Paraboloid**.
A very important surface, it's also known as a saddle surface, and it's the first shape we've described that can't be thought of as a surface of revolution, though, like the elliptic paraboloid, it is still a function.

Given by the equation $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$, it differs from the elliptic paraboloid by have one positive and one negative squared term. In one plane, the graph will have the trace of a parabola opening up, and in the other, a trace of a parabola opening down. The third plane, parallel to the function variable plane, will look like a hyperbola.

This surface is extremely important because it is the classic example of a graph where the derivatives are equal to zero, but where there is no extremum because the graph is going in different directions (up or down) depending on how you are traversing the surface.







You should practice drawing each of these graphs in various orientations to improve your drawing abilities (you'll need it), and getting a feel for the graphs. Also, think about what other surfaces might look like, such as cylinders or planes in three-space. Use visual aids to orient yourself. A slinky is great for thinking about helices cylinders. Consider creating 3-D models for problems involving intersecting planes to get used to rotating the graphs in your head. Experiment with graphing software. The better you are able to do this, the easier dealing with these functions will be in calculus. Formulas will help, but they will only get you so far.

**Practice Problems** (for 3-D surfaces):

1. For each of the basic 3-D surfaces listed in the handout, rewrite the formulas using cylindrical and spherical coordinates. Simplify as much as possible. For cylindrical, solve for z. For spherical, solve for ρ.
2. Use GraphCalc to graph one or two of the equations and reproduce on of the graphs in the handout. You will need to solve for z in rectangular coordinates. Some of these surfaces (nearly all of them) are not functions in the usual one-to-one sense, so you will need to graph the surface in two parts.
3. Some functions, like $y=x^2$, cannot be graphed as a 3-D surface in GraphCalc because it has no z variable. Two alternate strategies for graphing this function is to "swap" variables. Graph $z=x^2$, and rotate the graph so that the y-axis is in the position where the z-axis usually is. How is this graph different from the graph we really wanted? Another alternative is to solve for spherical coordinates and graph ρ from this equation. Explain which method you prefer and why.