Functions as Power Series

Our discussion of Geometric Power Series is rooted in a basic relationship between an infinite geometric series and the formula for its summation: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. In our case, we are going to reverse the relationship and consider functions with the form of the formula, and rewrite the function in the form of a summation: $\frac{a}{1-x} = \sum_{n=0}^{\infty} ax^n$, where x is standing in for our radius. However, we have to keep in mind that as a power series, we have to restrict our values for convergence. Our original function was defined on (-∞,1)U(1,∞), but the power series is defined only on the radius of convergence of the power series, which in the very general case is (-1,1). Despite this weakness, there are advantages to this representation. For one thing, the series is essentially a polynomial, and there are lots of things we can say about polynomials.

Example 1.

Let's consider two simple examples.

a. $f(x) = \frac{1}{1-4x^2}$. Write this function as a power series. Find the interval of convergence.

Here, $a = 1, r = 4x^2$. Thus $\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} 1\cdot\left(4x^2\right)^n = \sum_{n=0}^{\infty} 4^n x^{2n}$. The radius of convergence is when $|r| = 4x^2 < 1$. This occurs on $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We should check the endpoints, but neither of these converge.

b. $f(x) = \frac{3x}{6+7x}$. Write this function as a power series.

Here, in order to find the power series, we have to put it in the form $\frac{a}{1-r}$. What we are missing is that our constant in the denominator is not 1. So, divide everything by 6.

$f(x) = \frac{\frac{1}{2}x}{1-\left(-\frac{7}{6}x\right)}$. Now, we have that $a = \frac{1}{2}x, r = -\frac{7}{6}x$. Notice that since we had an addition sign in the denominator, our r value is negative. Thus, our power series is

$f(x) = \frac{3x}{6+7x} = \sum_{n=0}^{\infty} \frac{1}{2}x\cdot\left(-\frac{7}{6}x\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 7^n x^{n+1}}{2\cdot 6^n}$. The exponent of x is n+1 because we had x^n*x (from a), and we added exponents.

We mentioned above that the radius of convergence is highly restricted in the power series over that of the original function, but, we can rewrite our function so that we can center it around any point of interest.

Example 2.

Consider the function $f(x)=\frac{1}{1-x}$. This function, is centered at zero, and has an interval of convergence of (-1,1). Let's consider if we wanted to center this at 2, instead of zero. To shift x from zero to 2, we replace it with x-2, and with the leading negative, that becomes –(x-2). This means that we've added 2 to the denominator, and so we have to compensate for that, thus our denominator becomes (1-2)-(x-2) = -1-(x-2). We still need a leading 1, so we divide the whole function by -1. Thus, $f(x)=\frac{-1}{1+(x-2)}$ giving us $a=-1, r=-(x-2)=2-x$. Put that into the power series form we get $\sum_{n=0}^{\infty}-1\cdot(2-x)^n$. If we find the radius of convergence, $|2-x|<1 \Rightarrow -1<2-x<1 \Rightarrow -3<-x<-1$, so if we divide by -1, our interval of convergence is (1,3), which is centered at 2.

Our resulting power series then have the properties of series. We can easily add them. Indeed, for Example 1a, we could have factored the denominator and separated it into two simpler fractions by partial fractions, and obtained a power series by adding the two power series that result from each term.

Now that we have this way of creating power series, we can use integration or differentiation to create new series, or to allow us to represent more complicated functions as power series. We consider integration first.

Consider the function $f(x)=\ln x$. It's derivative is $f'(x)=\frac{1}{x}$ which is a rational function of the form we can represent as a power series. If we integrate the power series we find, we can create the power series for f(x). Indeed, we can perform a similar feat for $g(x)=\arctan x$, since $g'(x)=\frac{1}{1+x^2}$. For the logarithmic series, we will have to shift the center to x=1, because the function is not defined at x=0. We can do that as we did in Example 2. In Example 3, we will consider the inverse tangent case in detail.

Example 3.

Consider the function $g(x)=\arctan x$ and its derivative $g'(x)=\frac{1}{1+x^2}$. We can represent the derivative as the power series $g'(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n}$. Remember that the summation is just a series of terms, and we can integrate this term-by-term, or more directly, integrate the formula for the general term to derive the formula for the general term of the integral. In other words: $\int\sum_{n=0}^{\infty}(-1)^n x^{2n}dx=\sum_{n=0}^{\infty}\int(-1)^n x^{2n}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$. We can determine that C is 0 because arctan(0)=0. Thus the power series for the inverse tangent is $g(x)=\arctan(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$.

We can employ derivatives to find power series, too, for rational function with higher powers in the denominator by taking the derivative of the basic relationship we've been using: $\frac{a}{1-x} = \sum_{n=0}^{\infty} ax^n$. To make our procedure easier to follow, rewrite the left side as $a(1-x)^{-1} = \sum_{n=0}^{\infty} ax^n$. The table below gives the derivatives of both sides.

$$f(x) = a(1-x)^{-1} = \sum_{n=0}^{\infty} ax^n$$

$$f'(x) = (-1)(-1)a(1-x)^{-2} = \sum_{n=1}^{\infty} anx^{n-1}$$

$$f''(x) = (1)(-1)(-2)a(1-x)^{-3} = \sum_{n=2}^{\infty} an(n-1)x^{n-2}$$

$$f'''(x) = (1)(2)(-1)(-3)a(1-x)^{-4} = \sum_{n=3}^{\infty} an(n-1)(n-2)x^{n-3}$$

$$f^{IV}(x) = (1)(2)(3)(-1)(-4)a(1-x)^{-5} = \sum_{n=4}^{\infty} an(n-1)(n-2)(n-3)x^{n-4}$$

$$\vdots$$

$$f^{(k)}(x) = k!a(1-x)^{-(k+1)} = \sum_{n=k}^{\infty} an(n-1)(n-2)(n-3)...(n-k)x^{n-k}$$

So, we end up with a series of formulas for rational functions of any whole number power in the denominator. Our procedure will be to take the derivative of the standard geometric series formula until the left side has the correct power, and then put a and r into the equivalent formula on the right.

Example 4.

We will consider two example here: a simpler one, and a harder one.

a. For the function $g(x) = \frac{4}{(1-5x)^3}$, find the power series that can represent the function.

   Starting here: $f(x) = a(1-x)^{-1} = \sum_{n=0}^{\infty} ax^n$. Take the derivative twice to get here:

   $f''(x) = 2!a(1-x)^{-3} = \frac{2a}{(1-x)^3} = \sum_{n=2}^{\infty} an(n-1)x^{n-2}$. If the factorials get in the way of finding a, you can divide both sides of the equation by the factorial so that the numerator stands for a by itself. That would give us the equation $\frac{a}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{an(n-1)x^{n-2}}{2!}$. In this case, a=4, and our "r" value is 5x. Our equation then becomes:

$g(x)=\dfrac{4}{(1-5x)^3}=\sum_{n=2}^{\infty}2n(n-1)5^{n-2}x^{n-2}=\sum_{n=0}^{\infty}2(n+2)(n+1)5^n x^n$. We should then simplify, and, if we wish, we can also reset the summation to n=0, by replacing n with n+2. The advantages of this kind of representation are that if we needed to integrate, we can do so with this formula. This may not seem obviously helpful in this example, but the next example will be one that we can't easily integrate in the original form, and it can only be done with by parts or trig substitution and a lot of extra work.

b. For the function $h(x)=\dfrac{7x^2}{\left(1+3x^5\right)^4}$, we will need the third derivative of our base formula:

$3!a(1-x)^{-4}=\dfrac{3!a}{(1-x)^4}=\sum_{n=3}^{\infty}an(n-1)(n-2)x^{n-3}$, and dividing by the factorial, we get $\dfrac{a}{(1-x)^4}=\sum_{n=3}^{\infty}\dfrac{a}{3!}n(n-1)(n-2)x^{n-3}$. Thus here we have $a=7x^2$ and our "r" is $-3x^5$. We get $\dfrac{7x^2}{(1+3x^5)^4}=\sum_{n=3}^{\infty}\dfrac{7x^2}{3!}n(n-1)(n-2)\left(-3x^5\right)^{n-3}$. If we do some simplifying we get:

$h(x)=\dfrac{7x^2}{(1+3x^5)^4}=\sum_{n=3}^{\infty}(-1)^{n-3}\dfrac{7x^2}{2\cdot 3}3^{n-3}n(n-1)(n-2)x^{5(n-3)}=$

$\sum_{n=3}^{\infty}\dfrac{(-1)^{n-1}\cdot 7}{2}3^{n-4}n(n-1)(n-2)x^{5n-13}$. Some simplification points:

$(-1)^{n-3}=\dfrac{(-1)^{n-1}}{(-1)^2}=(-1)^{n-1}$. The exponent of x comes from

$x^2\cdot x^{5(n-3)}=x^2\cdot x^{5n-15}=x^{5n-15+2}=x^{5n-13}$. And the 3! reduces by $\dfrac{3^{n-3}}{2\cdot 3}=\dfrac{3^{n-3-1}}{2}=\dfrac{1}{2}\cdot 3^{n-4}$. As we mentioned above, this function would be tricky to integrate in its original form, but in the power series form it's quite straightforward. $\int\sum_{n=3}^{\infty}\dfrac{(-1)^{n-1}\cdot 7}{2}3^{n-4}n(n-1)(n-2)x^{5n-13}dx$. Recall that everything is considered a constant except for the x factor. We get:

$\sum_{n=3}^{\infty}\dfrac{(-1)^{n-1}\cdot 7}{2(5n-12)}3^{n-4}n(n-1)(n-2)x^{5n-12}+C$. If we are going to rewrite the equation with n=0, by replacing n with n+3 everywhere, it's easiest to do early on

$h(x)=\dfrac{7x^2}{(1+3x^5)^4}=\sum_{n=3}^{\infty}(-1)^{n-3}\dfrac{7x^2}{2\cdot 3}3^{n-3}n(n-1)(n-2)x^{5(n-3)}$ here to get

$\sum_{n=0}^{\infty}(-1)^n\dfrac{7x^2}{2\cdot 3}3^n(n+3)(n+2)(n-1)x^{5(n)}=\sum_{n=0}^{\infty}(-1)^n\dfrac{7}{2}\cdot 3^{n-1}(n+3)(n+2)(n+1)x^{5n+2}$, and

then our integral would be: $\sum_{n=0}^{\infty}\dfrac{(-1)^n\cdot 7}{2(5n+3)}3^{n-1}(n+3)(n+2)(n+1)x^{5n+3}+C$. Either answer is fine if the directions don't specify one form or the other.

Practice Problems:

Find the power series for each function. If a c-value is specified, center your interval of convergence there (if none is specified, assume c=0).

1. $f(x)=\frac{3}{2x-1}$ $\sum_{n=0}^{\infty}-3\cdot 2^n x^n$
2. $g(x)=\frac{3}{x+2}, c=-1$ $\sum_{n=0}^{\infty}\frac{3(-1)^n}{2^{n+1}}x^n$
3. $h(x)=\frac{3x}{2x^2+3x-2}$ (Hint: complete the square) $\sum_{n=0}^{\infty}\frac{-24}{25}\left[\frac{4}{5}\left(x+\frac{3}{4}\right)\right]^{2n}$
4. $F(x)=\frac{4}{4+x^2}$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}x^{2n}$
5. $G(x)=\ln(x), c=1$ $\sum_{n=0}^{\infty}\frac{-(1-x)^{n+1}}{n+1}$
6. $H(x)=\frac{2}{(x+1)^3}$ $\sum_{n=0}^{\infty}(n+2)(n+1)x^n$
7. $q(x)=\arctan\left(\frac{1}{2}x^2\right)$ $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{4}\right)^n\frac{x^{4n+2}}{4n+2}$
8. $r(x)=\frac{x(2+x)}{(1-x)^2}$ $\sum_{n=0}^{\infty}2x^{n+1}+\sum_{n=0}^{\infty}x^{n+2}$
9. $s(x)=\frac{9x^3}{\left(1+4x^3\right)^3}$ $\sum_{n=0}^{\infty}(-1)^n\frac{2}{9}4^n(n+2)(n+1)x^{3n+3}$
10. $t(x)=\frac{3}{2x-5}, c=4$ $\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n(x-4)^n$
11. $u(x)=\frac{11x}{2\left(1-\sqrt{x}\right)^3}$ $\sum_{n=0}^{\infty}\frac{11}{4}x^{\frac{n+2}{2}}$
12. $v(x)=\ln(x-4)$ $\sum_{n=0}^{\infty}-\left(\frac{1}{4}\right)^{n+1}\frac{x^{n+1}}{n+1}$