Sequences

Sequences are lists of numbers (separated by commas) that are arranged in a particular order. They need not have a clear relationship among the numbers in the sequence, but the "interesting" ones will. We call the nth element in the sequence a_n. There are two ways of defining a sequence:

- ✓ by means of a formula in terms of n, or
- ✓ by means of a formula in terms of previous elements in the sequence (recursively)
  - o key point on recursive sequences is that you need both a formula, and enough *seed* terms for the formula to be defined.

Example 1.

Find the first five terms of the sequence $a_n = 5 - \dfrac{1}{n^2}$.

$$a_1 = 5 - \frac{1}{1^2} = 4, a_2 = 5 - \frac{1}{2^2} = \frac{19}{4}, a_3 = 5 - \frac{1}{3^2} = \frac{44}{9}, a_4 = 5 - \frac{1}{4^2} = \frac{79}{16}, a_5 = 5 - \frac{1}{5^2} = \frac{124}{25}$$

Thus $4, \dfrac{19}{4}, \dfrac{44}{9}, \dfrac{79}{16}, \dfrac{124}{25}, \ldots$

Example 2.

Find the first five terms of the sequence $a_{n+1} = \dfrac{1}{3} a_n{}^2, a_1 = 9$.

$$a_1 = 9, a_2 = \frac{1}{3}(9)^2 = 27, a_3 = \frac{1}{3}(27)^2 = 243, a_4 = \frac{1}{3}(243)^2 = 19,683, a_5 = \frac{1}{3}(19,683)^2 = 129,140,163$$

Thus $9, 27, 243, 19683, 129140163, \ldots$ or $3^2, 3^3, 3^5, 3^9, 3^{17}, \ldots$

Note: This is the same sequence if we define $a_n = \dfrac{1}{3} a_{n-1}{}^2, a_1 = 9$. And it is usually possible to write the same sequence defined recursively but in terms of n instead. However, the resulting formula will depend on the seed value.

Both of the previous examples also highlight another difference between sequences: those that converge and those that don't. If we continue the sequence in Example 1, and particularly if we look at it as a decimal or a graph, the sequence terms approach the value of 5. We say this sequence **converges**. If we continue the sequence in Example 2, we can already see that the values are blowing up, and rapidly. We say that such a sequence **diverges**. Indeed, any sequence that *doesn't converge* is said to *diverge*.

One of the hardest problems with sequences, however, is when we are given the sequence and we are asked to come up with a rule for the pattern. This process requires developing an ability to recognize patterns, and the possible forms of common patterns. We should start with the simplest possible form of the pattern, and work up to more difficult options if those fail. When dealing with fractions, we can consider the numerator and denominator separately.

✓ Arithmetic sequences differ just by a constant additive value. For example: 3,5,7,9,11,... $2n+3$.

✓ Geometric Sequences differ by a constant multiple. For example: 3,6,12,24,48,... $3 \cdot 2^n$

✓ Factorials. For example: 1,1,2,6,24,120... $n!$

✓ Power series. For example: $1, \dfrac{1}{8}, \dfrac{1}{27}, \dfrac{1}{64}, \dfrac{1}{125}, ... \dfrac{1}{n^3}$

✓ Alternating signs. For example: 3,-5,7,-9,11,... $(-1)^n(2n+3)$

✓ Consider that your terms may have a common difference between one of these series, such as 4,19,44,79,124... $5n^2 - 1$ or $0, \dfrac{1}{7}, \dfrac{1}{26}, \dfrac{1}{63}, \dfrac{1}{124}, ... \dfrac{1}{n^3 - 1}$

✓ Consider terms may have been reduced.

✓ You may be able to develop a pattern from the first three terms of the sequence, but your proposed formula will have to successfully predict later terms because three terms is not really enough to establish a unique pattern.

Example 3.

Come up for a formula for the nth term of the sequence $1, -\dfrac{2}{3}, \dfrac{4}{9}, -\dfrac{8}{27}, ...$

o The alternating negative signs mean that we will need a $(-1)^n$ or a $(-1)^{n+1}$ term. If we start with n=0 (which is common), then if the first term is positive, $(-1)^0$ is positive, so use that; however, if the first term is negative, then $(-1)^{0+1}$ is negative, so use that version.

o Setting aside the negatives, there is no common difference between terms:
$1 - \dfrac{2}{3} = \dfrac{1}{3}, \dfrac{2}{3} - \dfrac{4}{9} = \dfrac{2}{9}$. So the series isn't arithmetic.

o The ratio between the terms is constant, however: $\left(\dfrac{2}{3}\right) \div (1) = \dfrac{2}{3}, \left(\dfrac{4}{9}\right) \div \left(\dfrac{2}{3}\right) = \dfrac{2}{3}$, so what we have is a geometric series. The common ratio is the exponential base.

o There is no extra multiplier for the sequence since $(2/3)^0 = 1$ which is the first term.

$$a_n = (-1)^n \left(\dfrac{2}{3}\right)^n = \left(-\dfrac{2}{3}\right)^n$$

Example 4.

Come up with a formula for the nth term of the sequence $1, x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}, ...$

o Let's consider the numerator and denominator separately. The numerator alone is the sequence $1, x, x^2, x^3, x^4, ...$ this is clearly x^n

o The denominator is the sequence $1,1,2,6,24,...$ if you write both the first two terms over 1. This is a sequence of factorials, n!, starting with n=0

o Put the two together. (be sure both your numerator and denominator terms start with the same value for n, either n=1 or n=0).

$$a_n = \dfrac{x^n}{n!}$$

Example 5.

Come up for a formula for the nth term of the sequence $\dfrac{3}{2},\dfrac{7}{4},\dfrac{15}{8},\dfrac{31}{16},\dfrac{63}{32},....$

- o Consider the numerator and denominator separately.
- o The denominator here is easiest: 2,4,8,16,32,...2^n, starting with n=1
- o The numerator 3,7,15,31,63,... doesn't have a common difference, nor does it have a common ratio, but the differences do show a pattern: 7-3=4, 15-7=8, 31-15=16, 63-31=32. This suggests that the sequence is a common difference away from the sequence 4, 8 16,32... which are powers of 2, or more specifically, 3=4-1, 7=8-1, 15=16-1, 31=32-1, 63=64-1. So our numerator sequence is then $2^{n+1}-1$. The exponent here is n+1 because our sequence starts with 4 rather than 2.
- o If we want our sequences to start with n=0 instead of n=1, we have to adjust n in our formula upwards by 1, i.e. 2^{n+1} and $2^{n+2}-1$. Then put everything together.

$$a_n = \dfrac{2^{n+2}-1}{2^{n+1}} = 2 - \dfrac{1}{2^{n+1}}$$

Practice Problems.

1. $3^2, 3^3, 3^5, 3^9, 3^{17},...$ (This is the same problem from Example 2. Come up with a non-recursive definition for a_n.)

2. $2,-1,\dfrac{1}{2},-\dfrac{1}{4},\dfrac{1}{8},....$

3. $1,6,120,5040,362880,....$

4. $0,1,3,6,10,15,21,....$

5. $5,10,20,40,...$

6. $\dfrac{7}{2},4,\dfrac{9}{2},5,...$

7. $0,\dfrac{1}{2},\dfrac{2}{3},\dfrac{3}{4},\dfrac{4}{5},...$

8. $2,\dfrac{1}{2},\dfrac{4}{3},\dfrac{3}{4},\dfrac{6}{5},...$ (Hint: subtract 1 from every term)

9. 0,1,0,-1,0,1,0,-1... (Hint: think trig functions.)

10. $1,2x,2x^2,\dfrac{4}{3}x^3,\dfrac{2}{3}x^4,\dfrac{4}{15}x^5,...$ (Hint: the fractions have been reduced. Compare to Example 4.)

Answers.

1. $a_n = 3^{2^n+1}, n = 0,1,2,3...$

2. $a_n = 2\left(-\dfrac{1}{2}\right)^n, n = 0,1,2,3..$

3. $a_n = (2n-1)!, n = 1,2,3..$

4. $a_{n+1} = a_n + n, a_0 = 0$

5. $a_n = 5(2^n), n = 0,1,2,3..$

6. $a_n = 3 + \dfrac{1}{2}n, n = 1, 2, 3..$

7. $a_n = \dfrac{n-1}{n}, n = 1, 2, 3..$

8. $a_n = 1 + (-1)^{n+1}\dfrac{1}{n}, n = 1, 2, 3..$

9. $a_n = \sin\left(\dfrac{n\pi}{2}\right), n = 0, 1, 2, 3..$ or $a_n = \cos\left(\dfrac{n\pi}{2}\right), n = 1, 2, 3...$

10. $a_n = \dfrac{(2x)^n}{n!}, n = 0, 1, 2, 3..$