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Series Tests

One of the major topics in Calculus is primarily just nasty algebra (and calculus only helps some of the time). Series are sums of terms (typically defined by a pattern, as in a sequence). When the sequence is finite, we can, at least in principle, always find the sum. When the series is infinite, some series will converge to a value, and some (most) will diverge to infinity. There are a number of possible ways to determine whether a series converges or diverges, and each test does not work (well) for every possible type of series. Our goal will not only be able to learn how to execute and interpret each test, but also to determine which test to use.

For each of the series tests we will consider the form of the series that applies to the test, the method of execution for each test, the conditions for convergence or divergence, and any other special properties that might apply.

Note: some series in our examples will start with n=0, and some with n=1. The distinction will not matter much for our tests unless otherwise noted. We will start with n=0 when our nth term is defined so that no term will result in division by 0. Terms that have factorials and exponentials in the denominators typically start with n=0, but power series often start with n=1 because using n=0 would cause the formula to look uglier to avoid division by zero.

Let's start with the simplest.

Example 1. nth-Term Test.

For any series $\sum_{n=0}^{\infty} a_n$ or $\sum_{n=1}^{\infty} a_n$, if the $\lim_{n \to \infty} a_n \neq 0$, the series will diverge.

- This test cannot be used to show convergence. Some series which do have a limit of 0 converge, and some will diverge.

  a. $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series. Even though $\lim_{n \to \infty} \frac{1}{n} = 0$, we will see with the Integral Test below that this series diverges.

  b. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a series where $\lim_{n \to \infty} \frac{1}{n^2} = 0$, and the series does converge. We can use the Integral Test or the p-test to see this, and we will reconsider this series again below.

  c. $\sum_{n=1}^{\infty} \frac{n!}{n^2}$ will diverge because if we look at the first couple terms of the sequence we can see that the $\lim_{n \to \infty} \frac{n!}{n^2} \neq 0$, i.e. $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{24}{5}, 20, ...$

- The nth-term test is a good test to start with, because it's easy, and if the series clearly diverges, there is no need to start with a more complicated calculation. If the nth term does go to zero, then move on to one of the other tests that are able to differentiate more finely.

Example 2. Alternating Series Test.

For an alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ (or a similar form, it does not matter whether the initial term is positive or negative, but here we are assuming that a_n is positive), if $\lim_{n \to \infty} a_n = 0$, the series will converge.

- Conditions that must apply: $a_n > a_{n+1} > 0$, i.e. the sequence is decreasing (you can use the first derivative test to determine this, or check graphically). It is this condition that assumes that a_n is positive; it will need to be for this condition to apply and the limit approach 0.
- We can estimate the sum of an alternating series in a way we can't for a non-alternating series. For the alternating series, the remainder is less than the value of the next term, i.e. $|R_N| \le a_{N+1}$. You can use this property to specify an error level and calculating, using the formula for a_N, the maximum number of terms you will need to take to get with the error value of the true (infinite) sum.
- This test is frequently confused with the nth term test, but more series will converge as an alternating series than will converge absolutely (when all terms are positive).

a. We said for the nth term test that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (we will prove it in a later example), but when we consider $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the Alternating Harmonic Series, will converge by the Alternating Series test. We have $a_n > a_{n+1} > 0$ because $\frac{1}{n} > \frac{1}{n+1} > 0$ because the denominator for n+1 is larger than n, and therefore, the fraction is smaller. And we have that $\lim_{n \to \infty}\frac{1}{n} = 0$, therefore the series converges. In addition, we have that $|R_N| \le \frac{1}{N+1}$, so if we want to estimate the sum and ensure that it's within 0.01 of the true sum, then $0.01 \le \frac{1}{N+1} \Rightarrow N+1 \le 100 \Rightarrow N \le 99$, you would need to add up no more than 99 terms. In practice, you may need less, but this number will guarantee your result.

b. $\sum_{n=1}^{\infty}\frac{(-3)^{n+1}}{n!}$ doesn't look like our traditional alternating series, but we can rewrite this series as $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^{n+1}}{n!}$. So, checking, we have $a_n > a_{n+1} > 0$ because $\frac{3^{n+1}}{n!} > \frac{3^{n+2}}{(n+1)!} > 0$ (factorials are increasing faster than powers of 3 as soon as you get past n=3), and $\lim_{n \to \infty}\frac{3^{n+1}}{n!} = 0$ (for the same reason). Therefore, the series converges.

c. $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^2}$ is tricky because the form of the equation is designed to confuse.

$(-1)^{2n+1} = (-1)^{2n}(-1) = ((-1)^2)^n(-1) = 1^n(-1) = (-1)$, which means the series can be

rewritten as $\sum\limits_{n=1}^{\infty}\dfrac{-1}{n^2}$, which is not an alternating series. The limit does converge to zero, but because it is not alternating, this test does not apply.

d. Some cosine and sine series can act like Alternating Series, as well. For instance $\sum\limits_{n=1}^{\infty}\dfrac{\cos(n\pi)}{n}$ or

$\sum\limits_{n-1}^{\infty}\dfrac{\sin\left(n+\dfrac{\pi}{2}\right)}{n}$, both have trigonometric components that reduce to (-1)^n or (-1)$^{n+1}$. Both series are just another way of writing the Alternating Harmonic series, (or it's negative version if the first term is negative, as it will be for cos(nπ). Indeed, if the argument of the trig function is any multiple of π/2, for either sine or cosine, you can rewrite the sequence as an alternating series, because any terms that are zero can be discarded. This behaviour is similar to what happens with any sine or cosine function, but you will need another test for the more general case.

Example 3. Telescoping Series Test

For a series of the form $\sum\limits_{n=1}^{\infty}a_n = \sum\limits_{n=1}^{\infty}\left(b_n - b_{n+1}\right)$, then the value of the sum will depend on the limit of b_n.

If $\lim\limits_{n\to\infty}b_n = L$, then the sum of the series is $S = b_1 - L$.

- o I confess, this was the one test I routinely would forget about.
- o The form of a_n is formed from two terms that are related to each other, and are the difference of two terms of a (different but related) sequence. Only the first and last terms matter because all the middle terms will cancel each other out.
- o Consider a telescoping series may be at work if you have a product of factors in the denominator such as n(n+1) or n(n+2), or n(n+3), or (n+1)(n+2), etc. (any factor that differences by a fixed constant). Apply partial fractions to determine b_n if possible (not all series of this form will work as the numerators will need to be equal and opposite for the cancelling effect to apply). If the factors differ by more than one (i.e. n(n+2) or (n+1)(n+3), etc.), then you will need more terms than b_1 to calculate the sum. However, the basic principle is the same.

a. The series $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)$ is a classic telescoping series. Consider the first couple terms

(unsimplified): $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)=\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\left(\dfrac{1}{4}-\dfrac{1}{5}\right)+\left(\dfrac{1}{5}-\dfrac{1}{6}\right)+...$ If we

rearrange the parentheses we have $\dfrac{1}{2}+\left(-\dfrac{1}{3}+\dfrac{1}{3}\right)+\left(-\dfrac{1}{4}+\dfrac{1}{4}\right)+\left(-\dfrac{1}{5}+\dfrac{1}{5}\right)+\left(-\dfrac{1}{6}+...\right)$, thus

except for the first term, all the remaining terms in the sequence are going to cancel out in the middle, and leave only the last term in the sequence. Which gives us

$\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)=\dfrac{1}{2}-\lim\limits_{n\to\infty}\dfrac{1}{n+1}=\dfrac{1}{2}-0=\dfrac{1}{2}$. It is important to note here that telescoping series will converge because the two terms are working in concert with each other. While it's

true that $\sum\limits_{n=1}^{\infty}\left(\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)=\sum\limits_{n=1}^{\infty}\dfrac{1}{n+1}-\sum\limits_{n=1}^{\infty}\dfrac{1}{n+2}$, this won't help us much because each

series is a version of the harmonic series, and will diverge individually. And what we learned when we were doing L'Hôpital's rule was that $\infty - \infty$ is an indeterminate form. As we can see here, they work together to converge, but that is by no means guaranteed by the form $\infty - \infty$.

b. The series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ is more typical of the form we see. If we use partial fractions, we can

rewrite the series as $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right)$. This is still a telescoping series, just not the simplest

case. To see how this works, let's consider the first several terms of the series (unsimplified):

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+3} \right) = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{9} \right) + \dots \quad \text{If we}$$

rearrange the terms as we did before:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \left(-\frac{1}{4} + \frac{1}{4} \right) + \left(-\frac{1}{5} + \frac{1}{5} \right) + \left(-\frac{1}{6} + \frac{1}{6} \right) - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} \dots \,. \quad \text{As you can see, all the terms in}$$

the middle cancel out, and leaves us with just the first three terms and the last three terms (because the difference between n+3 and n (the factors in our denominators) is three).

Therefore, the sum $S = b_1 + b_2 + b_3 - \lim_{n \to \infty} \left(b_{n-2} + b_{n-1} + b_n \right)$, which here is

$$S = 1 + \frac{1}{2} + \frac{1}{3} - \lim_{n \to \infty} \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \right). \quad \text{The limit of each one of those terms goes to zero, so}$$

$$S = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}. \quad \text{And remember that any time we have a means to calculate the actual sum,}$$

the series definitely converges to the value we find. For many tests, we can say they converge, but just not what to. We saw this with the Alternating Series test. All we had was an error formula. So, when we have a series of this form, this is a really powerful test. This method is not the only way to show that this series converges, but it is the only way to find the value of the infinite sum.

Example 4. P-series Test.

For a series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is some fixed constant, then the series will converge if p>1. If

$p \leq 1$, then the series will diverge.

o The p-series test is derived from the Integral Test, which we will describe later. P-series are quite common, so we use this test quite often, and don't re-derive it each time.

o We use the p-series test by itself to determine the convergence or divergence of simple series, and we use the results in combination with other tests to determine the convergence of more complicated series.

a. When $p \leq 0$, the result is actually extremely clear, the series will diverge. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{-1/2}}$ is

the same series as $\displaystyle\sum_{n=1}^{\infty} n^{1/2} = \sum_{n=1}^{\infty} \sqrt{n}$. In this case, each of the terms is actually getting bigger,

and by the nth term test, we know this series diverges.

b. When p>0, the behaviour of $0 < p \leq 1$ is different than the behaviour of p>1. The difference is the two series we considered under the nth term test: the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Even though a_n converges to zero in both cases, $\frac{1}{n}$ doesn't converge to zero fast enough and slowly increases to infinity, but $\frac{1}{n^2}$ does converge to zero fast enough. The p-series test is the quick way to recognize where the dividing line is.

Example 5. Integral Test.

For an series for the form $\sum_{n=1}^{\infty} a_n$ where a_n is defined by the function $f(n) \geq 0$, f must also be

continuous and decreasing, then $\int_{1}^{\infty} f(x)dx$ determines whether the series converges or diverges. If the improper integral diverges, then the series diverges; if the improper integral converges, then the integral converges. What's more, to estimate the series sum, the remainder R_N is bounded by

$0 < R_N < \int_{N}^{\infty} f(x)dx$. As we saw with the Alternating Series Test, we can use this formula to find a

maximum value for N so that the error is within a given value.

a. For the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$, we can use the Integral Test to prove that it fails to converge,

even though the limit of a_n goes to zero. $f(n) = \frac{1}{n}$, thus $\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} dx = \lim_{b \to \infty} \ln b = \infty$

(the lower limit, ln1=0).

b. For the p-series test, we have $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (for p≠1, which is the exception to the power rule and is

shown above). Integrate then $\int_{1}^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_{1}^{b} x^{-p} dx = \lim_{b \to \infty} \frac{x^{-p+1}}{-p+1}\Big|_{1}^{b} = \frac{1}{-p+1} \lim_{b \to \infty} b^{-p+1} - 1^{-p+1}$

and the only term that matters is $b^{-p+1} = \frac{1}{b^{p-1}}$. Recall that p is a fixed constant. if $-p+1 \geq 0$

(or p $\leq$ 1), then this term will go to 0 as b goes to ∞ since the large number will be in the denominator. If b goes to infinity in the numerator, it will blow up. This will be clearer if we consider two specific cases p = 1/3 and p=2.

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ we have $\int_{1}^{\infty} \frac{1}{x^{1/3}} dx = \lim_{b \to \infty} \int_{1}^{b} x^{-1/3} dx = \lim_{b \to \infty} \frac{3}{2} x^{2/3}\Big|_{1}^{b} = \frac{3}{2} \lim_{b \to \infty} b^{2/3} - 1^{2/3} = \infty$

d. For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we have $\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \int_{1}^{b} x^{-2} dx = \lim_{b \to \infty} \frac{x^{-1}}{-1}\Big|_{1}^{b} = -1 \cdot \lim_{b \to \infty} \frac{1}{b} - 1^{-1} = 1$.

e. The integral test also works for some other types of series besides power series. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$

consider the integral $\displaystyle\int_{1}^{\infty} x2^{-x}\,dx$. It needs to be integrated by parts to get

$$\lim_{b\to\infty}\int_{1}^{b} x2^{-x}\,dx = \lim_{b\to\infty}-\frac{1}{\ln 2}x2^{-x}-\frac{1}{(\ln 2)^2}2^{-x}\Bigg|_{1}^{b} = \lim_{b\to\infty}\frac{-b}{\ln 2\cdot 2^b}-\frac{1}{(\ln 2)^2 2^b}+\frac{1}{\ln 2\cdot 2}+\frac{1}{(\ln 2)^2 2}.$$

The only potentially problematic term is the first one $\dfrac{-b}{2^b}$ (ln2 doesn't matter), but by L'Hôpital's Rule, we can see that this term will converge to 0. Since the integral is finite, the series will converge.

o The Integral Test can't be used for series (at least not directly) that contain factorials since the function f(n) is not continuous: n! is not defined for values that are not non-negative integers. Other tests work better for such series. There is also a way to approximate n! with a crazy-looking exponential sequence for proof purposes, but we will not encounter this approximation in this course.

Example 6. Geometric Series.

A series of the form $\displaystyle\sum_{n=0}^{\infty} ar^n = a\sum_{n=0}^{\infty} r^n$ will converge whenever |r|<1. When |r|≥1, the series will diverge. Moreover, when the series does converge, the sum of the series is $S = \dfrac{a}{1-r}$.

o One important note is that the formula depends on the series beginning explicitly at n=0. If your series does not begin at n=0, you will need to rewrite your series so that it does, or else subtract off the missing terms for the sum of the series.

a. For the series $\displaystyle\sum_{n=0}^{n}\left(\frac{\pi}{4}\right)^n$, since π < 4, π/4 < 1; therefore, the series converges, and the sum of the

series is $S = \dfrac{1}{1-\dfrac{\pi}{4}} = \dfrac{1}{\dfrac{4-\pi}{4}} = \dfrac{4}{4-\pi} \approx 4.659...$

b. For the series $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{2}{e}\right)^n$, here it seems that a=3, but we are also missing the first term where

n=0. So if we rewrite the series to start at n=0, we have $\displaystyle 3\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^{n+1}$ (check, when n=0, the first

exponent is 1, when n=1, the first exponent is 2, etc. just as we had before). Getting rid of the

+1 in the exponent we get $\displaystyle 3\cdot\frac{2}{e}\sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n$. Now it's of the correct form to determine the sum. It

does not need to be exactly in this form to determine the convergence or divergence since while

the a value matters for the actual sum, only the r value determines the convergence. We have

that 2/e<1, so it converges, and it converges to $S = \dfrac{\dfrac{6}{e}}{1-\dfrac{2}{e}} = \dfrac{\dfrac{6}{e}}{\dfrac{e-2}{e}} = \dfrac{6}{e-2} \approx 8.35...$

c. By contrast, the series $\displaystyle\sum_{n=0}^{\infty}\left(\dfrac{\pi}{e}\right)^n$ diverges because π/e>0.

Example 7. Root Test.

The root test applies to a series of the form $\displaystyle\sum_{n=1}^{\infty} a_n$ (generally). The test says that if $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$ then the series converges, but if $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$ it diverges. But the test will be inconclusive if the same limit is equal to one.

- o The geometric series test can proved using the root test. The Root Test can be applied more generally, however. Good series to apply the root test do, however, do have exponents.
- o The Root Test and the Ratio Test have similar domains of application, but the algebra will be different. Which test you use will depend on which algebra will be easier. They are often inconclusive on the same series so in such cases, you will want to consider the direct comparison and limit comparison tests to resolve it. You can also try the integral test if there are no factorials, though the integral may be difficult.

a. For the series $\displaystyle\sum_{n=0}^{\infty}\left(\dfrac{\pi}{e}\right)^n$, the root test gives $\lim_{n\to\infty}\sqrt[n]{\left(\dfrac{\pi}{e}\right)^n} = \dfrac{\pi}{e} > 1$. Thus the series diverges, as we saw above.

b. For the series $\displaystyle\sum_{n=0}^{\infty}\dfrac{n}{(\ln n)^n}$, by the root test, we have $\lim_{n\to\infty}\sqrt[n]{\dfrac{n}{(\ln n)^n}} = \dfrac{\sqrt[n]{n}}{\ln n}$, since the ln(n) goes to infinity, but $\sqrt[n]{n}$ goes to 1, so the ratio is <1. Therefore, the series converges.

Example 8. The Ratio Test.

The Ratio test applies to series of the form $\displaystyle\sum_{n=1}^{\infty} a_n$ (generally). The test says that if $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$ then the series will converge. If the $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$ the series will diverge, and if the limit is equal to one, then the test is inconclusive.

- o The ratio test is commonly confused with the Limit Comparison Test (below).
- o This test works great with factorials and powers.
- o If the test is inconclusive, then options include the Integral Test, the direct comparison or the limit comparison.

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a. For the series $\sum_{n=1}^{\infty} \frac{n7^n}{n!}$, we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)7^{n+1}}{(n+1)!}}{\frac{n7^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)7^n \cdot 7}{n!(n+1)} \cdot \frac{n!}{n7^n}\right| = \lim_{n\to\infty}\left|\frac{7}{n}\right| < 1 \text{ for any n > 7. Thus, the series}$$

converges.

b. For the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$, we apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}3^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n 3^n}{n2^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n(-1)3^n \cdot 3}{(n+1)2^n \cdot 2} \cdot \frac{n2^n}{(-1)^n 3^n}\right| = \lim_{n\to\infty}\left|-\frac{3}{2} \cdot \frac{n}{n+1}\right| \text{ and since}$$

$$\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 1, \text{ the limit of our ratio test reduces to 3/2, which is}$$

greater than 1. Thus, the series diverges (absolutely). We might want to reconsider this problem as an alternating series, however, because for an alternating series, we just need to

show that $\lim_{n\to\infty}\left(\frac{3^n}{n2^n}\right)$ goes to 0. It does not in this case, but our test actually assumes that a_n is

positive and our test is for absolute convergence.

c. To test the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ with the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \lim_{n\to\infty}\left|\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = 1, \text{ and thus the test is inconclusive, and}$$

can be resolved by the Integral Test, or once derived, the p-test.

Example 9. Limit Comparison Test.

For a series of the form $\sum_{n=1}^{\infty} a_n$ we want to compare to a series of the form $\sum_{n=1}^{\infty} b_n$ which is typically

similar and whose behaviour we know from another test. Then if the ratio $\lim_{n\to\infty}\frac{a_n}{b_n} = L > 0$ (and finite)

and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ will also converge. On the other hand, if the ratio $\lim_{n\to\infty}\frac{a_n}{b_n} = L > 0$

(and finite) but $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ will also diverge.

- o The two series will do the same thing, so it doesn't help to choose another series which is inconclusive.
- o Do not confuse this test with the Ratio Test.
- o The form of the rule does not depend on the order of division. If you find the $\lim_{n\to\infty} \dfrac{b_n}{a_n} = M$, the same conditions will still apply since M here is equal to 1/L. You still can't get 0 or infinity as the limit. If you do, you didn't choose a good comparison series.
- o It is assumed, as with many of these rules, that $a_n, b_n > 0$, at least for some finite n that is large enough. If not, compare the absolute values, or use the alternating series test if possible.

a. For the series $\displaystyle\sum_{n=0}^{\infty} \dfrac{4^n}{3^n + 1}$ use the comparison series $\displaystyle\sum_{n=0}^{\infty} \dfrac{4^n}{3^n}$. Forming the ratio we have

$$\lim_{n\to\infty} \frac{\dfrac{4^n}{3^n+1}}{\dfrac{4^n}{3^n}} = \lim_{n\to\infty} \frac{4^n}{3^n+1}\cdot\frac{3^n}{4^n} = \lim_{n\to\infty}\frac{3^n}{3^n+1} = \lim_{n\to\infty}\frac{1}{1+\dfrac{1}{3^n}} = 1,$$ which is finite, and the two series

will behave the same. The series $\displaystyle\sum_{n=0}^{\infty}\dfrac{4^n}{3^n}$ diverges according to the geometric series test (or the root test), and thus the original series also diverges.

b. For the series $\displaystyle\sum_{n=1}^{\infty}\dfrac{\sqrt{n-5}}{n^2+1}$, we can compare it to the series $\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^{3/2}} = \sum_{n=1}^{\infty}\dfrac{\sqrt{n}}{n^2}$. Thus we have

$$\lim_{n\to\infty}\frac{\dfrac{\sqrt{n-5}}{n^2+1}}{\dfrac{\sqrt{n}}{n^2}} = \lim_{n\to\infty}\frac{\sqrt{n-5}}{\sqrt{n}}\cdot\frac{n^2}{n^2+1} = \lim_{n\to\infty}\sqrt{1-\frac{5}{n}}\cdot\frac{1}{1+\dfrac{1}{n^2}} = 1 \quad \text{By the p-series test, we know that}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2}$$ converges, and so therefore we know that $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n-5}}{n^2+1}$ also converges.

Examples 10. Direct Comparison Test.

The Direct Comparison Test is related to the squeeze theorem. If we have a series $\displaystyle\sum_{n=1}^{\infty} a_n$ that we wish

to determine the convergence of, and a comparison series $\displaystyle\sum_{n=1}^{\infty} b_n$ that we know converges, then if

$0 < a_n \le b_n$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. If the comparison series $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, and if

$b_n \le a_n$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ also diverges.

- o This test assumes that all terms of both comparison series are strictly positive, at least for some finite n. This test can be done absolutely by considering only the absolute values.
- o The inequalities here matter a lot. If you want can't get a simple series that is related in the appropriate way, then you will need to use the limit comparison, or some other test.

a. For the series $\sum\limits_{n=1}^{\infty}\dfrac{\sin n}{n^2+1}$, we can use the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ as our comparison series. Because

$\sin(n)\leq 1$ for all n. So $\dfrac{\sin n}{n^2}\leq\dfrac{1}{n^2}$, AND the denominator must be the same or **bigger** (to make

the whole term smaller), and that's the case here, so $\dfrac{\sin n}{n^2+1}\leq\dfrac{\sin n}{n^2}\leq\dfrac{1}{n^2}$. (Feel free to check

this intuition for $\dfrac{1}{n^2+1}\leq\dfrac{1}{n^2}$ for a couple terms to convince yourself.) We know from the p-

series test that $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ converges, and so therefore $\sum\limits_{n=1}^{\infty}\dfrac{\sin n}{n^2+1}$ must also converge.

b. In contrast, for the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2-1}$ we cannot use this test, because $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ converges but

$\dfrac{1}{n^2}\leq\dfrac{1}{n^2-1}$ which is not the relationship we need since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ converges. We should try

another test for this series.

c. For the series $\sum\limits_{n=1}^{\infty}\dfrac{n^2+1}{n^3}$, we choose the comparison series $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{n^3}=\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$. This is the Harmonic

series which we know diverges, so we want for $\dfrac{1}{n}\leq\dfrac{n^2+1}{n^3}$, which you can test for a few terms

to see that it is true. And so in this case, it the two series will diverge together.

d. In contrast, the series $\sum\limits_{n=1}^{\infty}\dfrac{n^2-1}{n^3-4}$ will need a different test to determine that it diverges.

Practice Problems.
Use the series tests to determine whether the series converges or diverges. Many of these problems
can be done more than one way. Bonus points for finding a second or third way to *correctly* test for
convergence or divergence. If you can find the sum of the series, do so (you won't be able to for all of
them). For #22 & #30, find the values for x for which the series will converge.

1. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n(n+1)}$

2. $\sum\limits_{n=1}^{\infty}\dfrac{2+\sin n}{n}$

3. $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n 2^{4n}}{(2n+1)!}$

4. $\sum\limits_{n=5}^{\infty}\dfrac{5}{n^2-3n-4}$

5. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n\sqrt{\ln n}}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n$

7. $\displaystyle\sum_{n=0}^{\infty} n^3 \left(\frac{7}{8}\right)^n$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left[2\cdot4\cdot6\cdot\ldots\cdot(2n)\right]}{2\cdot5\cdot8\cdot\ldots\cdot(3n-1)}$

11. $\displaystyle\sum_{n=1}^{\infty} \tan\frac{1}{n}$

12. $\displaystyle\sum_{n=1}^{\infty} 10(.25)^{n-1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n}$

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$

17. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right)$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^n}{(2n+1)!}$

19. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt[3]{n}}{\sqrt{n}}$

21. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n!}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^3-1}$

24. $\displaystyle\sum_{n=1}^{n} \sin 1$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$

27. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} e^{-n^{2}}$

28. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^{2}}{(3n)!}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n2^{n}}$

30. $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^{n}$