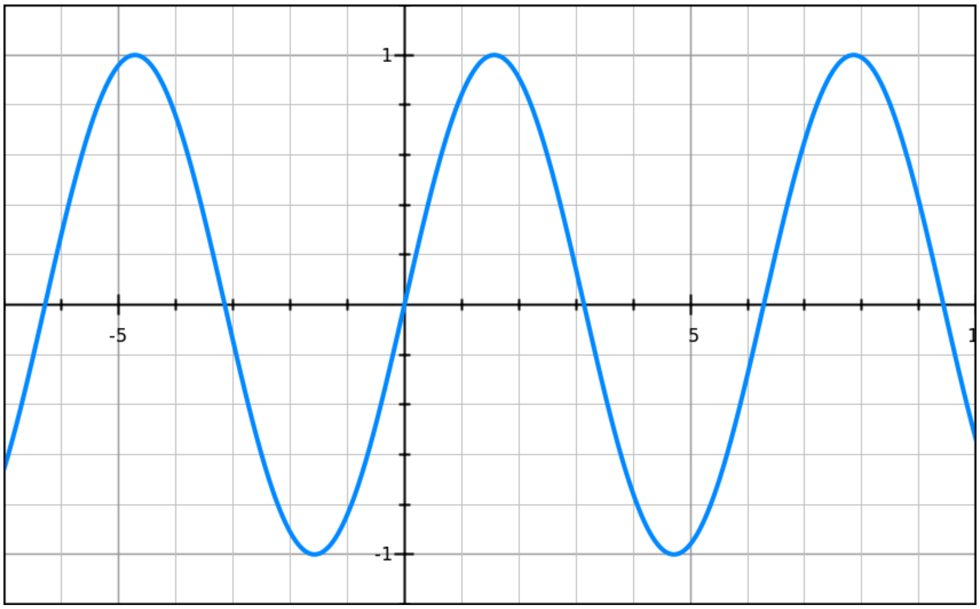
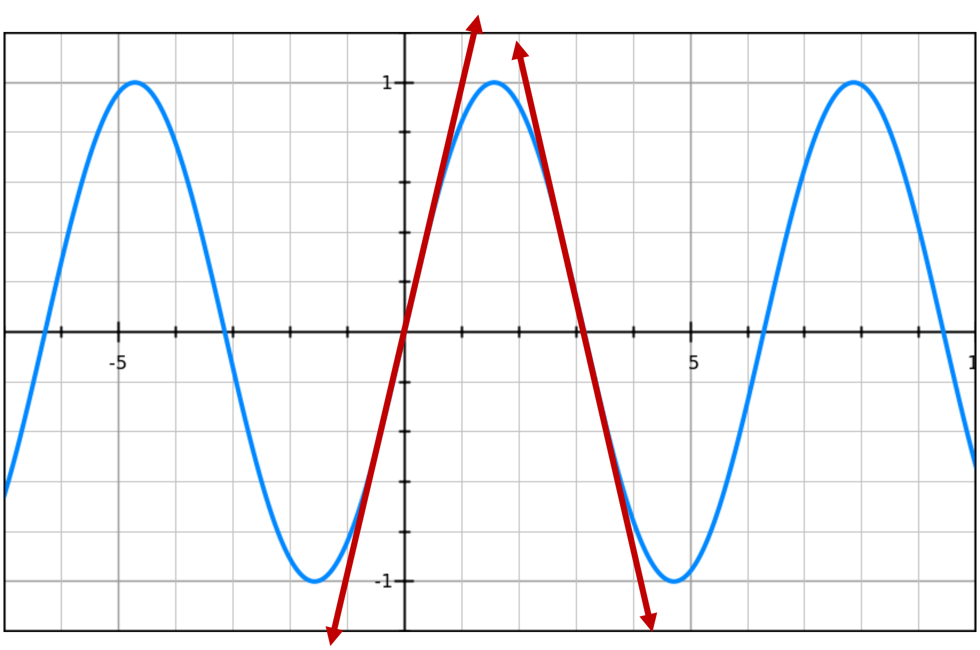
Exploring the Sine Function

Consider the graph of the sine function $f(x) = \sin(x)$ shown below.



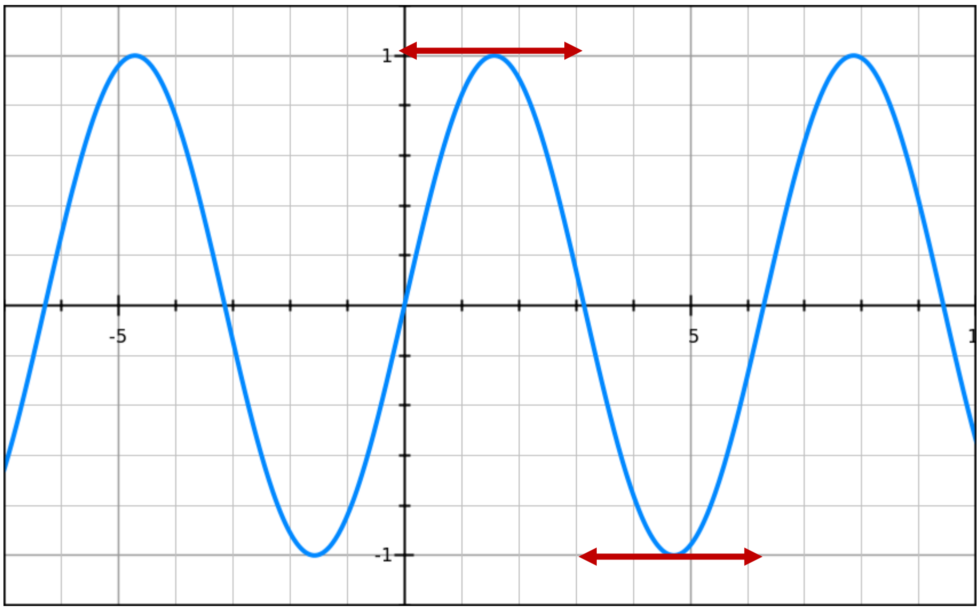
We'd like to find the derivative of this function, but before we find it algebraically, let's see if we can figure out what it should be from considering tangent lines to the graph at some key points.

The slope of the graph is steepest at $x = 0$, and $x = \pi$. What is the slope of the tangent line at $x = 0$? What is it at $x = \pi$?



This tells us two points on the graph of $f'(x)$. Namely, that $f'(0) = 1$, and $f'(\pi) = -1$. We can also see this pattern repeats every cycle, so that $f'(2\pi) = 1$ again and so on.

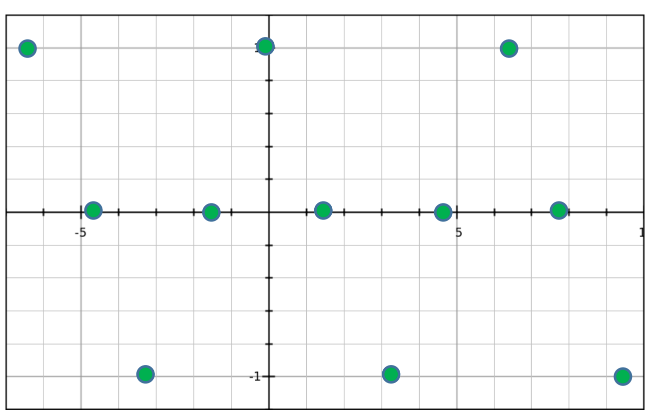
What about the slopes of the tangent lines at $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$?



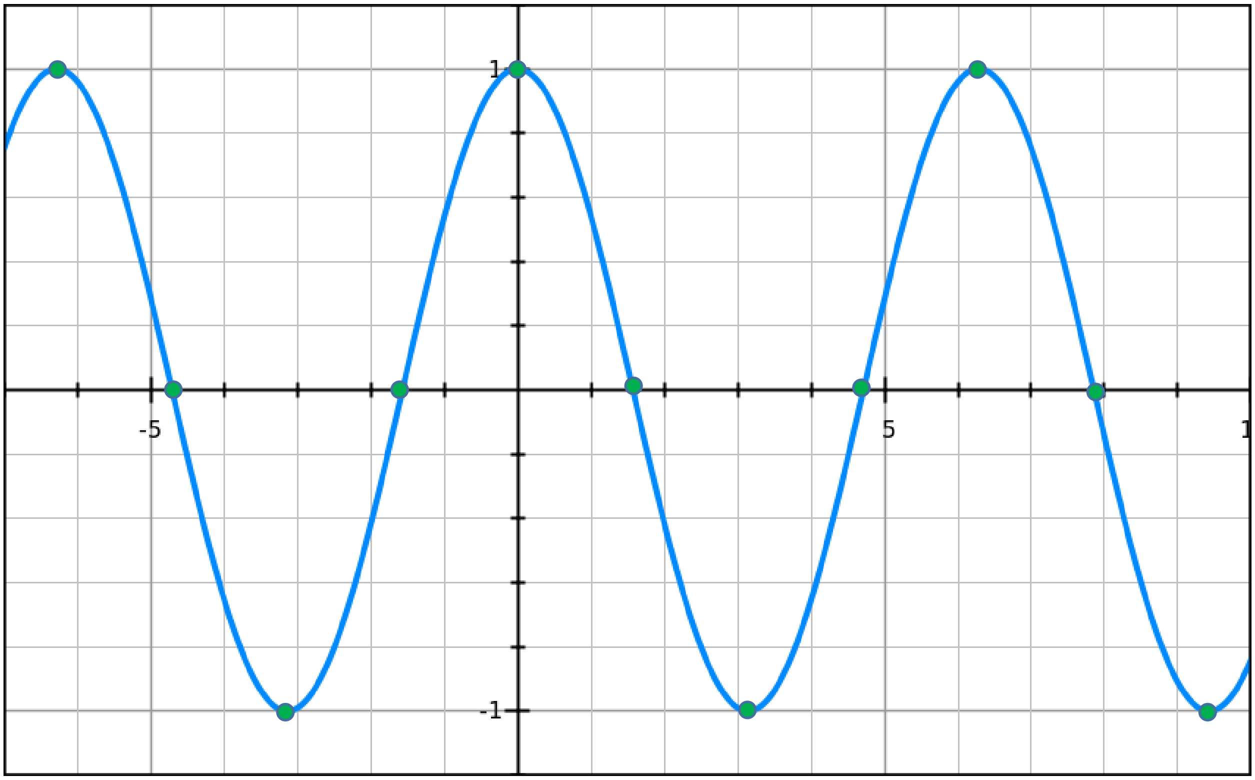
And these values repeat also, with the same slope at every odd multiple of π. So, below, we have a table of values for $f'(x)$. Let's fill in the values we found.

x	$f'(x)$
0	1
$\frac{\pi}{2}$	
π	-1
$\frac{3\pi}{2}$	
2π	1

A plot of the points is shown below. Can we think of any function that fits these values? Sketch it and see if you recognize it.



It’s the cosine function!



That seems to work, but how do we really prove it? We can using the definition of the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Plug in the sine function. Remember that $x + \Delta x$ both go inside the sine!

We should have

$$\lim_{\Delta x \to 0} \frac{sin(x + \Delta x) - sin(x)}{\Delta x}$$

In order to simplify the first function, we are going to need a trigonometric identity. This one:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

We can let $\alpha = x$, and $\beta = \Delta x$.

To reduce this and take the limit, we need to split this up a bit.

$$\lim_{\Delta x \to 0} \frac{sin(x)\cos(\Delta x) - sin(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta \mathrm{x})}{\Delta x}$$

On the left side, what happens to $\lim_{\Delta x \to 0} \cos(\Delta x)$?

As a result, what happens to the numerator on the left limit?

Now, let's look at the right limit.

$$\lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta \mathrm{x})}{\Delta x}$$

Since $\cos(x)$ doesn't depend on the variable in the limit, we can pull it outside the limit.

$$\cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta \mathrm{x})}{\Delta x}$$

Does anyone remember what we said in the last chapter was the value of this limit? $\lim_{\Delta x \to 0} \frac{\sin(\Delta \mathrm{x})}{\Delta x}$?

What does that leave us with a the derivative of $f(x) = \sin(x)$?