Doing Proofs with Trigonometric Identities

One of the main points of doing proofs with trigonometric identities is to help us practice manipulating equations and expressions utilizing various algebraic techniques and the myriad of trig identities we've collected so far. These skills will be essential when doing calculus.

1. **Some basic rules for proving identities**
   - Manipulate only one side of the equation at a time – this means no clearing denominators, or moving everything to one side to factor, or anything that you might do to an "equation". You can manipulate both sides of an equation and meet in the middle, but whatever you do, it can change only one side of the equation at a time.
   - Replacing terms with other known identities to simplify an expression is allowed
   - Be aware of the target you are trying to arrive at (the other side of the equation) and choose manipulations that seem like they will get you there.
   - It is often easier to choose the more complicated side of the equation to simplify, rather than trying to make the simpler side more complicated, since it can become more complicated in many more ways than the other side can be simplified.
   - Algebraic manipulations like finding a common denominator, multiplying by a quantity divided by itself, creating differences of squares, factoring (on one side only) are all acceptable techniques.
   - When using identities, try to relate trig functions that are naturally related to each other by other identities: sine and cosine, tangent and secant, cotangent and cosecant, or tangent and cotangent, etc.
   - Match all the angles in the expression if there is more than one to choose from.
   - When in doubt about what to do, convert everything to sine and cosine.
   - If it is possible to reduce a complicated (more than one term) denominator to a single term denominator, that is often helpful in cancelling or separating terms.

2. **Examples**

   a. $\frac{1-\sin\nu}{\cos\nu}+\frac{\cos\nu}{1-\sin\nu}=2\sec\nu$

   Difference of Squares. There are four trig functions that can be defined as a difference of squares: $\sin^2\theta=1-\cos^2\theta$, $\cos^2\theta=1-\sin^2\theta$, $\tan^2\theta=\sec^2\theta-1$, $\cot^2\theta=\csc^2\theta-1$. All of these come from the Pythagorean identities. Therefore, it is possible to reduce denominators of the form $1\pm\sin\theta$, $1\pm\cos\theta$, $1\pm\sec\theta$, $1\pm\csc\theta$, or any of these in the reverse order, by means of multiplying by the missing factor needed to create the difference of squares (we can think of it as the "conjugate", since like complex conjugates, the terms are the same, but only the sign between them is different). We can then reduce the denominator to a single term, and further simplification can proceed from there. Consider what happens when we apply this technique to this problem. *(Incidentally, the variable in this problem is the Greek letter nu. Yes, it is rather difficult to tell nu from the English v. Thus, never use both in the same equation!)*

$$\frac{1-\sin\nu}{\cos\nu}+\frac{\cos\nu}{1-\sin\nu}\cdot\frac{1+\sin\nu}{1+\sin\nu}=$$
$$\frac{1-\sin\nu}{\cos\nu}+\frac{\cos\nu(1+\sin\nu)}{1-\sin^2\nu}=\frac{1-\sin\nu}{\cos\nu}+\frac{\cos\nu(1+\sin\nu)}{\cos^2\nu}$$
$$\frac{1-\sin\nu}{\cos\nu}+\frac{1+\sin\nu}{\cos\nu}=\frac{1+1-\sin\nu+\sin\nu}{\cos\nu}$$
$$\frac{2}{\cos\nu}=2\sec\nu$$

Notice how easily the problem reduced once we simplified the denominator to a single term. Also notice one other trick was employed here that is quite common: do distribute in the difference of square part of the fraction, so that an identity can be applied, but do not distribute in the other part of the fraction. Distributing will only have to be undone through factoring in order for the cancelling to take place.

In this problem we employed only two identities: the Pythagorean identity for sine and cosine, and also that the reciprocal of cosine is secant. Everything else was just algebra.

Because this problem had two fractions, you can achieve the same results simply by finding a common denominator, however, that technique will not work in the cases where you have only one fraction. In the case of denominators with $1\pm\tan\theta$ or $1\pm\cot\theta$ which do not have identities involving a *difference* of squares (rather, they have a *sum* of squares), other techniques will have to be employed since the difference of squares technique can't be used to reduce the denominator.

b. $\sec^4\chi-\sec^2\chi=\tan^4\chi+\tan^2\chi$

Pythagorean identities. In many problems, you will have two different trig functions at work. If these trig identities come in the pairs associated with the Pythagorean identities, we should be able to employ those identities to reduce the problem in a straight-forward way. If the two functions are not pairs associated with Pythagorean identities, we should convert everything to sine and cosine. This last approach will often require more algebra, and so is to be avoided whenever possible. Here, secant and tangent are associated by the identity: $1+\tan^2\theta=\sec^2\theta$. Because our identity employs squares rather than 4th powers, factoring will allow us to apply the identity more easily. *(Incidentally, the variable used here is the Greek letter chi. As with nu, unless you can differentiate it in your handwriting, don't use this in the same equation as an English x.)*

$$\sec^4\chi-\sec^2\chi=\sec^2\chi(\sec^2\chi-1)=$$
$$(1+\tan^2\chi)\tan^2\chi=\tan^4\chi+\tan^2\chi$$

Of course, it should go without saying that in order to apply the identities correctly, there must be a squared term in order to make the replacement. The last thing you want to do is introduce square roots. If no squares are present, revert to the default strategy: convert to sine and cosine.

c. $\frac{1+\sin\gamma}{1-\sin\gamma}=(\sec\gamma+\tan\gamma)^2$

Work from the more complicated side. It's often a tendency for us, because we read from left-to-right, to start proofs also on the left side. Sometimes, this can be the harder route. Not impossible, but perhaps less well motivated, or less clear as to why we did something. In a problem like this one, it may be much clearer if we start from one side. In this problem, we are also working with two functions on one side, and a third on the other. On the left side we have only sine functions, so figuring out how to generate sine functions may be more difficult than reducing the other side. We can actually do it using one of the techniques I've mentioned previously, but that would defeat the point of this example! *(Incidentally, the variable used here is a Greek gamma, not a y!)*

$$(\sec\gamma+\tan\gamma)^2=\left(\frac{1}{\cos\gamma}+\frac{\sin\gamma}{\cos\gamma}\right)^2=$$

$$\left(\frac{1+\sin\gamma}{\cos\gamma}\right)^2=\frac{(1+\sin\gamma)^2}{\cos^2\gamma}=$$

$$\frac{(1+\sin\gamma)^2}{1-\sin^2\gamma}=\frac{(1+\sin\gamma)^2}{(1-\sin\gamma)(1+\sin\gamma)}=$$

$$\frac{1+\sin\gamma}{1-\sin\gamma}$$

Notice that the only identities I used here were for secant and tangent in terms of sine and cosine, and the Pythagorean identity in the denominator. I also did not multiply out the numerator and just left it squared so that I could cancel later. This seemed like a useful thing to do because the target form had this form already. Multiplying it out could have been done later if necessary, but if I do it without thinking about it, I may just have to undo it, and make the problem look much worse than necessary.

d. $\frac{\sin^3\xi+\cos^3\xi}{1-2\cos^2\xi}=\frac{\sec\xi-\sin\xi}{\tan\xi-1}$

Do you remember how to factor? How about complex fractions? No matter what we do with this problem, we are going to have to work with some pretty tricky algebra. If we work with just the right side, we are going to have to work with complex fractions. If we work with the left side, we are going to have to factor a sum of cubes. It's also not clear which side of the equation we should start with. This is one of those examples where we can work both sides of the equation (separately!) in the hope of reaching a common somewhere in the middle. Undoing the steps on one side gives the required proof. We will use this method to review the algebra involved on both sides. *(The Greek letter used here is xi, and notoriously difficult one that, sadly, even many math professors can't write correctly.)*

$$\frac{\sin^3\xi+\cos^3\xi}{1-2\cos^2\xi} = \frac{\sec\xi-\sin\xi}{\tan\xi-1}$$

$$\frac{(\sin\xi+\cos\xi)(\sin^2\xi-\sin\xi\cos\xi+\cos^2\xi)}{1-2\cos^2\xi} = \frac{\frac{1}{\cos\xi}-\sin\xi}{\frac{\sin\xi}{\cos\xi}-1}$$

$$\frac{(\sin\xi+\cos\xi)(1-\sin\xi\cos\xi)}{1-2\cos^2\xi} = \frac{\frac{1}{\cos\xi}-\sin\xi}{\frac{\sin\xi}{\cos\xi}-1}\cdot\frac{\cos\xi}{\cos\xi}$$

$$= \frac{1-\sin\xi\cos\xi}{\sin\xi-\cos\xi}$$

$$= \frac{1-\sin\xi\cos\xi}{\sin\xi-\cos\xi}\cdot\frac{\sin\xi+\cos\xi}{\sin\xi+\cos\xi}$$

$$= \frac{(1-\sin\xi\cos\xi)(\sin\xi+\cos\xi)}{\sin^2\xi-\cos^2\xi}$$

$$\frac{(\sin\xi+\cos\xi)(1-\sin\xi\cos\xi)}{-(2\cos^2\xi-1)} = \frac{(1-\sin\xi\cos\xi)(\sin\xi+\cos\xi)}{-(\cos^2\xi-\sin^2\xi)}$$

$$\frac{(\sin\xi+\cos\xi)(1-\sin\xi\cos\xi)}{-\cos(2\xi)} = \frac{(1-\sin\xi\cos\xi)(\sin\xi+\cos\xi)}{-\cos(2\xi)}$$

The solution on the right factors and reduces the second factor with the Pythagorean identity and then stops. You should be reluctant to just use a double angle formula (or any other formula) unless there is a clear reason to do so. On the right side, there is more obvious work to do. Writing as just sine and cosine, and reducing the complex fraction. At that point, we have one matching factor with the left side, but we are missing one. So the question I have to ask myself is, if I introduce missing factor (into both the numerator and the denominator) on the right, will that help me match the very odd looking denominator? As it turns out, it does, because it simplifies to one of the other identities for the cosine double angle formula. Remember that there are three versions! That justifies introducing the identity, however, it could be by using the Pythagorean identity on the sine-squared function on the right instead. However, now that we've met in the middle, you can see that if you follow the steps in a U-shape from either side, you can create a proof working from just one side to get to the other side. It is not, however, always clear why, for instance, you are switching from cosine double-angle formulas to another cosine double-angle formula, whereas it is clearer in the two-column procedure. Notice that only simplifying was permitted and each side was manipulated independently, never in unison.

e. $\cot(2\upsilon)=\frac{1}{2}(\cot\upsilon-\tan\upsilon)$

Double-Angle Formulas. Only use double-angle formulas in problems in problems (as above) when they are necessary to equate parts of a problem, or when one side of an identity employs a double angle, but the other side does not. In this particular problem, it's not entirely clear which side to start on. The cotangent double-angle formula can be arrived at from cosine and sine double-angle formulas respectively. If we start on the right side, we should expect to be looking for sine and cosine double angle formulas so that we can switch over. *(The variable used here is the Greek letter upsilon.)*

$$\frac{1}{2}(\cot\upsilon - \tan\upsilon) = \frac{1}{2}\left(\frac{\cos\upsilon}{\sin\upsilon} - \frac{\sin\upsilon}{\cos\upsilon}\right) =$$
$$\frac{1}{2}\left(\frac{\cos\upsilon}{\sin\upsilon}\cdot\frac{\cos\upsilon}{\cos\upsilon} - \frac{\sin\upsilon}{\cos\upsilon}\cdot\frac{\sin\upsilon}{\sin\upsilon}\right) =$$
$$\frac{1}{2}\left(\frac{\cos^2\upsilon}{\sin\upsilon\cos\upsilon} - \frac{\sin^2\upsilon}{\cos\upsilon\sin\upsilon}\right) =$$
$$\frac{1}{2}\left(\frac{\cos^2\upsilon - \sin^2\upsilon}{\sin\upsilon\cos\upsilon}\right) = \left(\frac{\cos^2\upsilon - \sin^2\upsilon}{2\sin\upsilon\cos\upsilon}\right) =$$
$$\frac{\cos(2\upsilon)}{\sin(2\upsilon)} = \cot(2\upsilon)$$

Our procedure here was to convert everything to sine and cosine, followed by finding a common denominator. Followed by the double-angle formulas we spoke about above. If we had worked in the opposite order, after applying the double-angle formulas, we would have separated the fraction into two terms and reduced. The same rules apply to half-angle formulas.

f. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

Sum and Difference Formulas. The only time you should be using sum and difference formulas, for the most part, is when you have two different angles, or, if you have odd multiples of individual angles. For instance, 3θ, can be broken down into θ+2θ. That's the only way to reduce it to angles for which we have other identities. The only thing to do with these problems is to apply the identities and plow through the algebra. *(The Greek letters here are alpha and beta.)*

$$\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} =$$
$$\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} =$$
$$\tan\alpha + \tan\beta$$

Here, all we did was apply the sine identity, and separate the terms, and reduce. Product-to-sum formulas are not covered in this course, but they are encountered briefly in Calculus.

These are variations on the sum and difference formulas, so the same rules apply with these.

Problems.
Prove the identities.

i. $(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)=1$

ii. $3\sin^2\kappa+4\cos^2\kappa=3+\cos^2\kappa$

iii. $1-\dfrac{\sin^2\delta}{1-\cos\delta}=-\cos\delta$

iv. $\dfrac{\cos\eta+1}{\cos\eta-1}=\dfrac{1+\sec\eta}{1-\sec\eta}$

v. $\dfrac{\sin\rho}{\sin\rho-\cos\rho}=\dfrac{1}{1-\cot\rho}$

vi. $\dfrac{\cot\lambda}{1-\tan\lambda}+\dfrac{\tan\lambda}{1-\cot\lambda}=1+\tan\lambda+\cot\lambda$

vii. $\csc\varphi-\cot\varphi=\dfrac{\sin\varphi}{1+\cos\varphi}$

viii. $\dfrac{\tan\omega-\cot\omega}{\tan\omega+\cot\omega}+1=2\sin^2\omega$

ix. $\dfrac{1+\sin\psi}{1-\sin\psi}-\dfrac{1-\sin\psi}{1+\sin\psi}=4\tan\psi\sec\psi$

x. $\dfrac{\sin^3\tau+\cos^3\tau}{\sin\tau+\cos\tau}=1-\sin\tau\cos\tau$

xi. $\dfrac{(2\cos^2\varepsilon-1)^2}{\cos^4\varepsilon-\sin^4\varepsilon}=1-2\sin^2\varepsilon$

xii. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\tan\beta$

xiii. $\sin\left(\dfrac{3\pi}{2}+\vartheta\right)=-\cos\vartheta$

xiv. $\sec(\mu+\nu)=\dfrac{\csc\mu\csc\nu}{\cot\mu\cot\nu-1}$

xv. $\dfrac{\sin(3\sigma)}{\sin\sigma}-\dfrac{\cos(3\sigma)}{\cos\sigma}=2$

xvi. $\cos\zeta=\dfrac{1-\tan^2\left(\dfrac{\zeta}{2}\right)}{1+\tan^2\left(\dfrac{\zeta}{2}\right)}$

xvii. $\sin^2\varsigma\cos^2\varsigma=\dfrac{1}{8}\left[1-\cos(4\varsigma)\right]$