Eigenvalues & Eigenvectors

Eigenvalues are a very important concept in linear algebra, and one that comes up in other mathematics courses as well. The word "*eigen*" is German for "inherent" or "characteristic", and so what we are talking about here are "characteristic constants and vectors" associated with a given matrix, or in broader terms, any operator. In linear algebra, eigenvalues are the values of λ that satisfies the equation

$$A\vec{x} = \lambda\vec{x}$$

The **A** is an nxn matrix, $\vec{x}$ is a vector, and λ is a constant. We'll see examples where λ is real, and others where it's complex. In all of our examples with start with an A matrix that is initially filled with real entries, but everything we do will also apply to matrices with complex entries at the start.

To find out what λ is for a given matrix, we need to solve this equation. We start by putting everything on one side and factoring out the $\vec{x}$.

$$A\vec{x} - \lambda\vec{x} = 0 \rightarrow (A - \lambda I)\vec{x} = 0$$

We factor out the $\vec{x}$ on the right because matrix multiplication is not commutative. And we insert the I for the identity matrix because we can't subtract a constant from a matrix, and the I does not change the equation anywhere else.

This equation is now homogeneous with (A-λI) a matrix. We are looking for the values of λ and the vectors $\vec{x}$ that will give a non-trivial solution to the system, thus, we'll need to find the null space of the matrix A-λI.

In principle, this is simple, but let's do a couple calculations and see how it works in detail.

Example 1. Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$.
We start by finding the matrix A-λI: $A - \lambda I = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$.

Notice that this step just involves subtracting λ's from the diagonal of the matrix.

For there to be a null space, the determinant of the matrix must be equal to zero, and that will give us conditions on λ.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - (3)(2) = 0$$

FOILing out this equation in λ gives us:

$$(3-\lambda)(4-\lambda) - (3)(2) = 0 \rightarrow 12 - 3\lambda - 4\lambda + \lambda^2 - 6 = \lambda^2 - 7\lambda + 6 = 0$$

This result is called the **characteristic equation** of the matrix. We now factor and solve for λ. If the equation cannot be factored, use the quadratic formula instead.

$$\lambda^2 - 7\lambda + 6 = 0 \rightarrow (\lambda - 6)(\lambda - 1) = 0$$

Our eigenvalues for this matrix are $\lambda_1 = 6 \; and \; \lambda_2 = 1$. It doesn't matter what order you notate them in, but each eigenvalue will go with a different eigenvector and the subscripts will help us keep track of which vector goes with which value of λ.

Now we have found the eigenvalues that will give us a non-trivial null space, but we still need to find the vectors that will satisfy the equation. We need to plug in each λ value into A-λI separately and find the null space in each case.

For $\lambda_1 = 6$, A-λI = $\begin{bmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3-6 & 2 \\ 3 & 4-6 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$.

We can see by inspection in a 2x2 matrix that the system is dependent since the equations are multiples of each other. We just need to take one of them and for the homogeneous system, set it equal to zero.

$$3x_1 - 2x_2 = 0 \rightarrow 3x_1 = 2x_2 \rightarrow x_1 = \frac{2}{3}x_2$$

With x_2 free. The solution to the system then is $\vec{x} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} x_2$. We can choose x_2 to make these whole numbers if we wish. If x_2=3, then $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Or we could normalize the vector to make it unit length $\vec{v_1} = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$. However, most of the time we will just stick to whole numbers as those are the most convenient to work with much of the time, and any multiple of this vector is correct. We can adjust for different types of vectors if a problem calls for it.

Let's continue to find the vector that goes with the other value of λ.

For $\lambda_2 = 1$, A-λI = $\begin{bmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3-1 & 2 \\ 3 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

As before, we have an obviously dependent system. For larger matrices, you'll have to row reduce to reduced echelon form, but for 2x2 just choose one of the equations to solve. It doesn't matter which one.

$$2x_1 + 2x_2 = 0 \rightarrow 2x_1 = -2x_2 \rightarrow x_1 = -x_2$$

So the solution to the system is $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$, and thus the eigenvector here is $\vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Let's try a 3x3 matrix. We generally have to choose special matrices because the factoring is so much more difficult unless we choose extremely carefully, particularly if we hope to have real eigenvectors. And what's worse, **doing row operations on a matrix changes the eigenvalues**, so we can't use that to simplify the procedure any.

Example 2. Find the eigenvalues and eigenvectors for $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 6 & 4 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & 6 & 4-\lambda \end{bmatrix}$$

And so the determinant, expanding along the middle row is:

$$(-1-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 0 & 4-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda)(4-\lambda) = 0$$

This is our characteristic equation, and since it's already factored, we have eigenvalues of $\lambda_1 = -1, \lambda_2 = 3, and\ \lambda_3 = 4$. An nxn matrix can have up to n distinct eigenvalues.

Replace the λ's one at a time and find the null space of each A-λI matrix.

$$\begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & 6 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3-(-1) & 1 & 2 \\ 0 & -1-(-1) & 0 \\ 0 & 6 & 4-(-1) \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 6 & 5 \end{bmatrix}$$

The reduced echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 7/24 \\ 0 & 1 & 5/6 \\ 0 & 0 & 0 \end{bmatrix}$. The parametric solution of this matrix $\vec{x} = \begin{bmatrix} -7/24 \\ -5/6 \\ 1 \end{bmatrix} x_3$. If we choose x_3 to be 24, we'll get whole numbers for $\vec{v_1} = \begin{bmatrix} -7 \\ -20 \\ 24 \end{bmatrix}$.

Repeating for the other two.

$$\begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & 6 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3-3 & 1 & 2 \\ 0 & -1-3 & 0 \\ 0 & 6 & 4-3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -4 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

The reduced echelon form of this matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The parametric solution of this matrix $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$. Thus $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

And

$$\begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & 6 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3-4 & 1 & 2 \\ 0 & -1-4 & 0 \\ 0 & 6 & 4-4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -5 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

The reduced echelon form of this matrix is $\begin{bmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$. The parametric solution of this matrix $\vec{x} = \begin{bmatrix}2\\0\\1\end{bmatrix}x_3$. Thus $\vec{v_2} = \begin{bmatrix}2\\0\\1\end{bmatrix}$.

The set of solutions to the eigenvector equation is called the **eigenspace**.

Example 3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix}1 & -3\\-4 & 1\end{bmatrix}$.
Sometimes the eigenvalues are real, but not pretty.

$$\det(A - \lambda I) = \begin{vmatrix}1-\lambda & -3\\-4 & 1-\lambda\end{vmatrix} = (1-\lambda)(1-\lambda) - (-3)(-4) = 0$$

$$1 - \lambda - \lambda + \lambda^2 - 12 = 0 \rightarrow \lambda^2 - 2\lambda - 11 = 0$$

Use the quadratic formula to solve.

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} = \frac{2 \pm \sqrt{48}}{2} = 1 \pm 2\sqrt{3}$$

As before, we have two eigenvalues and we plug them back in one at a time to find the eigenvectors.

For $\lambda_1 = 1 + 2\sqrt{3}$, A-λI = $\begin{bmatrix}1-\lambda & -3\\-4 & 1-\lambda\end{bmatrix} = \begin{bmatrix}1-(1+2\sqrt{3}) & -3\\-4 & 1-(1+2\sqrt{3})\end{bmatrix} = \begin{bmatrix}-2\sqrt{3} & -3\\-4 & -2\sqrt{3}\end{bmatrix}$

It's harder to see with the radicals involved, but these still are dependent equations. In order to avoid dividing by a negative, use the bottom equation.

$$-4x_1 + (-2\sqrt{3})x_2 = 0 \rightarrow 3 - 4 = 2\sqrt{3}x_2 \rightarrow x_1 = \frac{-\sqrt{3}}{2}x_2$$

So the solution to the system is $\vec{x} = \begin{bmatrix}\frac{-\sqrt{3}}{2}\\1\end{bmatrix}x_2$, and thus the eigenvector here is $\vec{v_1} = \begin{bmatrix}-\sqrt{3}\\2\end{bmatrix}$. There isn't really any way to make this "nice".

For $\lambda_2 = 1 - 2\sqrt{3}$, A-λI = $\begin{bmatrix}1-\lambda & -3\\-4 & 1-\lambda\end{bmatrix} = \begin{bmatrix}1-(1-2\sqrt{3}) & -3\\-4 & 1-(1-2\sqrt{3})\end{bmatrix} = \begin{bmatrix}2\sqrt{3} & -3\\-4 & 2\sqrt{3}\end{bmatrix}$.

When a square root is involved, we can cut down on the number of steps to find the second eigenvector. The solution will just be the conjugate of the first one. Compare the equations in the second row of the A-λI matrix. The only difference is the sign of the square root term. Thus, the solution to the system is $\vec{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix} x_2$, and thus the eigenvector here is $\overrightarrow{v_1} = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix}$. We'll do the same trick when we do a complex example.

Example 4. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.
This matrix, we can see by inspection, doesn't have an inverse since its determinant is already zero. What this tells us upfront is that one eigenvalue of the matrix is 0 since it already has a null space. But we'll do the procedure just as we've done above to find the other one.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - (2)(0) = (1-\lambda)(-\lambda) = 0$$

The eigenvalues are 1 and 0.

The 0 eigenvalue satisfies the original matrix, so we need to find that null space.

$$x_1 + 2x_2 = 0 \rightarrow x_1 = -2x_2$$

So the solution to the system is $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$, and thus the eigenvector here is $\overrightarrow{v_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For the second eigenvalue we get: $\begin{bmatrix} 1-1 & 2 \\ 0 & 0-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$. Using the last equation $-x_2 = 0$ and x_1 is free. So the solution to the system is $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1$, and thus the eigenvector here is $\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let's do one with a complex eigenvalue.

Example 5. Find the eigenvalues for the matrix $A = \begin{bmatrix} -2 & 2 \\ -5 & 0 \end{bmatrix}$.
Begin the problem the same as before.

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ -5 & -\lambda \end{vmatrix} = (-2-\lambda)(-\lambda) - (2)(-5) = \lambda^2 + 2\lambda + 10 = 0$$

This equation doesn't factor, so we go to the quadratic formula.

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

Plugging in the positive eigenvalue, we get a matrix with complex values.

For $\lambda_1 = -1 + 3i$, A-λI = $\begin{bmatrix} -2-\lambda & 2 \\ -5 & 0-\lambda \end{bmatrix} = \begin{bmatrix} -2-(-1+3i) & 2 \\ -5 & -(-1+3i) \end{bmatrix} = \begin{bmatrix} -1-3i & 2 \\ -5 & 1-3i \end{bmatrix}$

Just like in the real eigenvalue cases, these two equations are dependent. If you set your calculator to handle imaginary values, it can row-reduce the matrix for you, but there is no need for that in the 2x2 case. Just take one of the equations and solve it as before. The bottom one will require the least algebra since we won't have to rationalize any complex numbers in the denominator.

$$-5x_1 + (1-3i)x_2 = 0 \to -5x_1 = -(1-3i)x_2 \to x_1 = \frac{1-3i}{5}x_2$$

So the solution to the system is $\vec{x} = \begin{bmatrix} -\frac{1}{5} + \frac{3}{5}i \\ 1 \end{bmatrix} x_2$, and thus the eigenvector here is $\overrightarrow{v_1} = \begin{bmatrix} -1+3i \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} i$. Some applications want the real part of these complex vectors and the imaginary parts, so we often write the complex vectors in this form rather than as a single vector.

As with the real case involving square roots, the eigenvector resulting from the other eigenvalue is the conjugate of the first, so $\overrightarrow{v_2} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} i$.

Practice Problems.
Find the eigenvalues and eigenvectors for each of the problems below. Use correct subscripting notation to indicate which eigenvalues are paired with which eigenvectors.

1. $\begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 2 & 9 \\ 1 & 10 \end{bmatrix}$
3. $\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$
5. $\begin{bmatrix} 4 & 5 \\ 6 & 11 \end{bmatrix}$
6. $\begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix}$
7. $\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$
8. $\begin{bmatrix} -2 & 5 \\ 7 & 0 \end{bmatrix}$
9. $\begin{bmatrix} -3 & 7 \\ 5 & -1 \end{bmatrix}$
10. $\begin{bmatrix} -4 & 1 \\ 6 & -5 \end{bmatrix}$
11. $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$
12. $\begin{bmatrix} -4 & 5 \\ -5 & -4 \end{bmatrix}$
13. $\begin{bmatrix} -2 & 2 \\ -5 & 6 \end{bmatrix}$
14. $\begin{bmatrix} -2 & 5 & 3 \\ 0 & 2 & -4 \\ 0 & -1 & 2 \end{bmatrix}$
15. $\begin{bmatrix} 4 & -2 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 5 \end{bmatrix}$