Homework #4 Math 268 Spring 2012

1a. linearly independent & spans $\mathbb{R}^3$

b. linearly independent & spans $\mathbb{R}^3$

c. linearly independent; does not span $\mathbb{R}^3$

d. not linearly independent; spans $\mathbb{R}^3$

2a. Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -6 \\ 9 \end{bmatrix} \right\}$

b. Span $\left\{ \begin{bmatrix} -3 \\ 2 \\ 6 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -9 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

both a, b span a 4-dimensional subspace. remove any vector & check that you now get an independent set.

I removed v_3 & v_4 respectively.

3. b. false. a basis must both span and linearly independent.

a. true

c. true

d. false. only "as large as possible" in the sense of still being linearly independent.

e. true. you can use row operations to test dependency of a set, but it does not tell you which vector is the problem.

f. true

g. true

3h. false.

i. false. one determines the pivots from B but use the corresponding columns from A.

j. true

k. false $P_B [x]_B = \vec{x}$

l. false. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$

m. true

n. true (it must pass through the origin)

o. true

p. false. it equals the number of columns

q. true

r. true

s. true

t. false (they are C-coordinate vectors of the vectors in B)

u. true

4. a. $P_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$ $\quad P_B [x]_B = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \vec{x} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

b. $P_B = \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad P_B [x]_B = \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \vec{x} = \begin{bmatrix} 8 \\ -5 \\ -3 \end{bmatrix}$

c. $P_B = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$ $\quad P_B^{-1} \vec{x} = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = [x]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

d. $P_B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 3 & 8 & 3 \end{bmatrix}$ $\quad P_B^{-1} \vec{x} = \begin{bmatrix} 2 & 1/2 & -1/2 \\ -3/2 & 0 & 1/2 \\ 2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

4e. $P_B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ $\quad P_B[x]_B = \vec{x}$

$[x]_B = P_B^{-1}\vec{x}$

$P_B^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 5 \end{bmatrix}$

f. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$P_B^{-1}\vec{x} = [x]_B \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

5a. $p\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix} + q\begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -5 \\ 2 \\ 6 \end{bmatrix}$ $\quad B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 2 \\ 6 \end{bmatrix} \right\}$

dimension $= 3$

b. $B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \right\}$ dimension $= 3$

Spans $\mathbb{R}^3$

c. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ -2 \\ 1 \end{bmatrix} \right\}$ basis for Col A

dimension 4, spans $\mathbb{R}^4$

A rowreduces further to

$$\begin{bmatrix} 1 & 2 & -4 & 0 & 0 & 23 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

free free free

$x_1 = -2x_2 + 4x_3 - 23x_6$
$x_2 = x_2$
$x_3 = x_3$
$x_4 = 3x_6$
$x_5 = -4x_6$
$x_6 = x_6$
$x_7 = 0$

$$\vec{x} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} -23 \\ 0 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

basis for Nul A $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -23 \\ 0 \\ 0 \\ 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$ has dimension $= 3$

5d. $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\-4\\1\\0\end{bmatrix}, \begin{bmatrix}6\\-18\\9\\-1\end{bmatrix} \right\}$ basis for P_3
dimension 4

6. Col A = $\left\{ \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\\-2\\-2\end{bmatrix}, \begin{bmatrix}-2\\-3\\0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\\-3\\2\end{bmatrix}, \begin{bmatrix}-2\\-3\\6\\0\\-1\end{bmatrix} \right\}$ dimension = 5
= rank A

Row A = $\left\{ \begin{bmatrix}1\\1\\-2\\-4\\1\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\\-3\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\-13\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \right\}$ dimension = 5

dimension Nul A = 1

B reduces to $\begin{bmatrix}1&0&0&-3&0&0\\0&1&0&1&0&0\\0&0&1&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$
free

$x_1 = 3x_4$
$x_2 = -1x_4$
$x_3 = -1x_4 \Rightarrow \vec{x} = x_4 \begin{bmatrix}3\\-1\\-1\\1\\0\\0\end{bmatrix}$
$x_4 = x_4$
$x_5 = 0$
$x_6 = 0$

7.a. rank 2 means 2 pivots   dim Nul A + rank A = n
n here is 7, so dim Nul A = 5
dim Row A = 2 since dim Row A = Rank A

A^T also still has 2 pivots so rank A = rank(A^T) = 2

b. dim Nul A + dim Col A = n   n = 8 here so since dim Col A = # of pivots.

dim Nul A = 4

Col A $\neq \mathbb{R}^4$. Col A is 4-dimensional, but it is a subspace of $\mathbb{R}^6$

c. dim Nul $A = 5$ dim Nul A + dim Col $A = n$, here $n = 7$

So dim Col $A = 2$

d. Largest possible # of pivots is 4, so largest dim Col A is also 4

e. dim Nul A + dim Col A = n the largest size of Col A will gives the smallest dim of Nul A. we can have up to 5 pivots (the size of the smallest dimension of the matrix) but since n is also 5, smallest dim of Nul A = 0

f. (for a hint to this, compare w/ Nul A of Question #6)
yes. if we have 5 equations & 6 unknowns, then the A matrix is 5x6. one dimension of the null space leaves 5 pivots. Since $\mathbb{R}^m$ (the range) is equal to $\mathbb{R}^5$, 5 linearly independent vectors are a basis for the space. A is onto and Col A forms the basis.

8. a. $\underset{C\leftarrow B}{P} = \begin{bmatrix} -2 & 4 \\ 3 & 6 \end{bmatrix}$ $\begin{bmatrix} -2 & 4 \\ 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

b. $\left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 2 & 1 & 8 & -7 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 9 & -8 \\ 0 & 1 & -10 & 9 \end{array}\right]$ $\begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix} = \underset{C\leftarrow B}{P}$

$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 1 \\ 8 & -7 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 9 & 8 \\ 0 & 1 & 10 & 9 \end{array}\right]$ $\begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix} = \underset{B\leftarrow C}{P}$

c. $\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ -3 & 0 & 2 \end{bmatrix} = \underset{D\leftarrow F}{P}$ $\underset{D\leftarrow F}{P}^{-1} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/4 \\ -7/8 \\ 5/8 \end{bmatrix}$

8d. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$\underset{C\leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ $\quad \underset{C\leftarrow B}{P}^{-1} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/28 \\ -11/28 \\ 13/28 \end{bmatrix}$