Linear Transformations

The definition of a linear transformation is as follows:

$T: x \longmapsto Ax$ is a linear transformation from $\vec{x} \in \mathbb{R}^n$ to $A\vec{x} \in \mathbb{R}^m$ if the following conditions hold for every vector $\vec{u}$ *and* $\vec{v}$ in $\mathbb{R}^n$ and real scalar c.

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2) $T(c\vec{u}) = cT(\vec{u})$
3) $T(\vec{0}) = \vec{0}$

This is the version of the definition for a matrix, but the same three conditions hold for any linear transformation, whether or not it's a matrix. There are many things to say about linear transformations, but one of the key things we need to be able to do is to check that a transformation is linear. To do that, we'll check the three conditions.

Technically, the third condition is entailed by the first, but it can helpful to check the zero condition because if the transformation is going to fail, this will be the easiest one see a problem in.

Example 1. Determine if $T(\vec{x}) = A\vec{x}$ is linear for the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
We need to check the transformation on generic vectors, like $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. We can't choose specific values because we have to show that the transformation is linear for all values of possible pairs of vectors.

$$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} u_2+v_2 \\ u_1+v_1 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$

The second condition checks as follows.

$$T(c\vec{u}) = A(c\vec{u}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} cu_2 \\ cu_1 \end{bmatrix}$$

$$cT(\vec{u}) = c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} cu_2 \\ cu_1 \end{bmatrix}$$

Finally, we can use the fact that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$ to check the last condition. If we let $u_1 = 0$ and $u_2 = 0$, we get the zero vector on both sides of the equation as required.

Example 2. Determine if $f(x) = ax + b$ is a linear transformation. Assume that b≠0.

To determine if this is a linear transformation, we need to start checking the conditions. The easiest one to check is the third condition. Is it true that f(0)=0? No. Since f(0)=a(0)+b=b. Given the condition that b≠0, this is not a linear transformation.

Example 3. Determine if the derivative operator is a linear transformation.
The derivative operator $\frac{d}{dx}$ operates on functions. So let's use general functions, call them f(x) and g(x). To determine whether derivatives represent a linear transform, all we need are the properties we already know about derivatives.

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

This is one of the properties of derivatives we learned in calculus and it satisfies condition one of our definition.

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$$

This is another property of derivatives we learned in calculus and it satisfies the requirements of condition two.

$$\frac{d}{dx}[0] = 0$$

The derivative of zero is zero. So the third condition checks as well. That means that the derivative does satisfy the conditions of a linear operator.

Example 4. Determine if the integration operator satisfies the definition of a linear transformation on the set of functions whose anti-derivative F(x)=0 at x=0.

Consider the function defined by F(x)=$\int_0^x f(t)dt$. According to the rules of definite integrals $\int_0^x f(t)dt = F(x) - F(0)$. Since F(0) is defined to be zero, we just get F(x). Let's check the integral for the properties of a linear transformation.

$$\int_0^x [f(t)+g(t)]dt = \int_0^x f(t)dt + \int_0^x g(t)dt$$

This satisfies the first condition of the linear transformation.

$$\int_0^x kf(t)dt = k\int_0^x f(t)dt$$

This is satisfies the second condition of the linear transformation.

$$\int_0^x 0 dt = 0$$

All these properties are true for all functions, not just those whose antiderivatives are zero at zero.

Example 5. Consider the transformation defined as $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ |x_1| - x_3 \\ 1 \end{bmatrix}$

The easiest condition to check is to show that $T(\vec{0}) \neq 0$, since if we let $x_1 = x_2 = x_3 = 0$ we end up with the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The absolute value will also cause problems. To prove that a linear transformation fails it can be helpful to pick specific values to try to force the linearity to fail. Only when you are proving it is true in all cases do you have to depend on the generic vectors or functions.

Suppose we changed the definition to $S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ |x_1| - x_3 \\ 0 \end{bmatrix}$. Now the zero vector is okay. Which vectors could we pick to show that it still fails the definition? In the sum property, choose two vectors whose sums add to zero.

$$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Applying $S(\vec{u} + \vec{v}) = S(\vec{0}) = \vec{0}$, but $S(\vec{u}) + S(\vec{v}) = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \neq \vec{0}$

We can also get the absolute value component to fail if we use a negative scalar on the second property.

$S(-1\vec{v}) = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$, but $-1S(\vec{v}) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, which aren't the same.

You only need to get one of them to fail, not all of them, and sometimes it takes creativity to choose the right vectors to create a counterexample.

Practice Problems.

Determine if the following transformations are linear. If they are, prove it by checking the three conditions in the definition. If they are not, find a counterexample that makes one or more of the properties fail.

1. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 4x_2 \\ -x_3 \\ 2x_3 \end{bmatrix}$
2. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 - x_2 \\ x_2^2 + 5x_1 \\ 0 \end{bmatrix}$
3. Gradient operator ∇ on functions of 3 variables, f(x,y,z) and g(x,y,z).
4. $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

5. $A = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ for general θ.
6. Integration operator $\int_0^x \int_0^y f(u,v)dudv$.
7. $A = \begin{bmatrix} -2 & 0 & 1 \\ 3 & -1 & 5 \\ 2 & -4 & 0 \end{bmatrix}$
8. The transformation that maps (x,y) in rectangular coordinates unto (r,θ) in polar coordinates.