Name KEY

Math 285, Final Exam, Spring 2012

Instructions: Show all work. You may use a TI-89 to check your answers, but points will be deducted if you obtain answers solely from your calculator. Be sure to answer all parts of all the questions, and use exact answers unless directed to round, or in word problems.

1. Classify the following differential equations as linear or nonlinear, ordinary or partial, and their order. (6 points each)
   a. $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$ nonlinear, ordinary, 2nd order
   b. $\frac{\partial^2 y}{\partial y \partial x} + 2x\frac{\partial^3 y}{\partial x^3} = 0$ linear, partial, 3rd order
   c. $y' + (\sin t)y = y^{-2}$ nonlinear, ordinary, first order
   d. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = u_t$ linear, partial, 4th order

2. Determine which method should be used to solve the following equations. Choose from exact, linear (give the integrating factor), separable, homogeneous (state the degree), Bernoulli (state n). Do not solve the equations, just determine the method to be used. (7 points each)
   a. $(4+t)y' + 3ty = (4+t)$

      $y' + \frac{3t}{4+t}y = 1$ linear integrating factor μ

   b. $y' - 7x^2y = x^2y^5$

      Bernoulli $n = 5$
      (this is also separable, but the integration is nasty that way)

   c. $y' = \frac{x^3+y^3}{2xy}$

      homogeneous (order = 1)

   d. $\sin y\, dx + y\cos(x+3)\, dy = 0$ Separable

      $\sin y\, dx = -y\cos(x+3)dy$

      $\frac{dx}{\cos(x+3)} = \frac{-y}{\sin y}dy$

3. Determine if the following differential equations can be solved by separation of variables. Show work to justify your answer if you wish to receive partial credit. (8 points each)

a. $u_{xx} + u_{xt} + u_t = 0$ $\quad u = XT \quad u_{xx} = X''T \quad u_{xt} = X'T' \quad u_t = XT'$

$$X''T + X'T' + XT' = 0$$
$$X''T = -(X' + X)T' \implies \frac{X''}{X'+X} = -\frac{T'}{T} \quad \text{yes}$$

b. $u_{xx} + xyu_{yy} = 0$ $\quad u = XY \quad u_{xx} = X''Y \quad u_{yy} = XY''$

$$X''Y + xyXY'' = 0$$
$$X''Y = -xyXY'' \implies \frac{X''}{xX} = -\frac{yY''}{Y} \quad \text{yes}$$

c. $u_{xx} + u_{yy} + xu = 0$ $\quad u = XY \quad u_{xx} = X''Y \quad u_{yy} = XY''$

$$X''Y + XY'' + xXY = 0$$
$$XY'' = -(X'' + xX)Y \implies \frac{X}{X''+xX} = -\frac{Y}{Y''} \quad \text{yes}$$

4. For each pair of solution below and the stated forcing function, state the Ansatz you'd use for the method of undetermined coefficients. Or state that variation of parameters would work better and why. (8 points each)

#	y_1	y_2	g(t)	Ansatz Y(t)
a	e^{-2t}	te^{-2t}	e^{-t}	Ae^{-t}
b	$e^{-t}\sin t$	$e^{-t}\cos t$	$t^2 + e^{-t}$	$At^2 + Bt + C + De^{-t}$
c	sin(2t)	cos(2t)	5cos(2t)	$At\cos 2t + Bt\sin 2t$

5. Solve the differential equation and find the general solution $y' + 4ty = (1-2t)e^{-2t}$ for the initial condition y(1)=0. (25 points)

$$\mu = e^{\int 4t\,dt} = e^{2t^2}$$

$$e^{2t^2}y' + 4te^{2t^2}y = (1-2t)e^{-2t}e^{2t^2}$$

$$\left(e^{2t^2}y\right)' = (1-2t)e^{-2t+2t^2}$$

$$u = -2t + 2t^2$$
$$du = -2 + 4t\,dt$$
$$= -2(1-2t)\,dt$$
$$-\tfrac{1}{2}du = (1-2t)\,dt$$

$$e^{2t^2}y = \int (1-2t)e^{-2t+2t^2}\,dt$$

$$e^{2t^2}y = \int -\tfrac{1}{2}e^{u}\,du$$

$$\frac{e^{2t^2}y}{e^{2t^2}} = \frac{-\frac{1}{2}e^{-2t+2t^2} + C}{e^{2t^2}}$$

$$y = -\tfrac{1}{2}e^{-2t} + Ce^{-2t^2}$$

$$0 = -\tfrac{1}{2}e^{-2} + Ce^{-2} \qquad C = \tfrac{1}{2}$$

$$\boxed{y = -\tfrac{1}{2}e^{-2t} + \tfrac{1}{2}e^{-2t^2}}$$

6. Solve the initial value homogeneous second order differential equations with constant coefficients $y''+2y'-35y=0, y(0)=1, y'(0)=0$. Be sure to clearly indicate the characteristic equation in each case. Solve for any variables. (15 points)

$r^2+2r-35=0$ characteristic equation

$(r+7)(r-5)=0$

$r=-7, r=5$

$y=Ae^{-7t}+Be^{5t}$

$1=A+B \quad B=1-A$

$y'=-7Ae^{-7t}+5Be^{5t} = -7Ae^{-7t}+(5-5A)e^{5t}$

$0=-7A+5-5A \Rightarrow -12A+5=0 \quad A=5/12 \quad B=1-5/12=7/12$

$$y=\frac{5}{12}e^{-7t}+\frac{7}{12}e^{5t}$$

7. Solve the boundary value problem $y''+4y=0, y(0)=0, y(\pi)=0$. Is the solution unique? (15 points)

$r^2+4=0$

$r=\pm 2i$

$y=A\cos 2x+B\sin 2x$

$0=A(1)+B(0) \Rightarrow A=0$

$0=B\sin 2\pi$ $\quad$ $\sin 2\pi$ is already 0 so B is undetermined

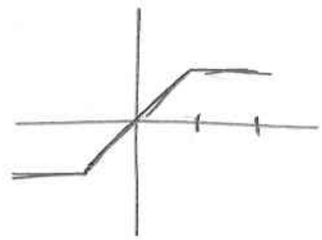
$\therefore$ The solution

$y=B\sin 2x$ is not unique.

8. Extend the function $f(x) = \begin{cases} x, 0 < x < 1 \\ 1, 1 \le x < 2 \end{cases}$ as the indicated type of Fourier series. *State the piecewise definition of the periodic piece on the appropriate interval (-L,L)* and state the resulting period. Sketch a graph of the resulting extension. Do not attempt to calculator the Fourier series itself. (10 points each)

a. As a sine (only) series

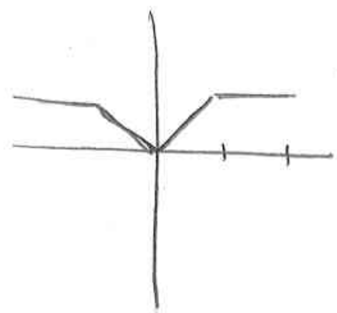
$$f(x) = \begin{cases} -1 & -2 < x < -1 \\ x & -1 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$$



period = 4

b. As a cosine (only) series

$$f(x) = \begin{cases} 1 & -2 < x < -1 \\ -x & -1 \le x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$$



period = 4

or $f(x) = \begin{cases} -1 & -2 < x < -1 \\ |x| & -1 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$

c. Set the interval (-L,0) to be zero. What kind of Fourier series would result?

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$$

period = 4

both sine & cosine terms

9. Solve the wave equation $a^2u_{xx} = u_{tt}$ for a=1, and given the boundary conditions $u(x,0) = \begin{cases} 0, & 0 < x < 3 \\ x-1, & 3 \le x < 6 \\ 0, & 6 \le x < 9 \end{cases}$, $u(0,t) = 0, u(9,t) = 0$. (40 points) $u_t(x,0) = 0$ ← missing condition

$X(0)=0 \quad X(9)=0 \qquad T'(0)=0$

$u = XT \quad u_{xx} = X''T \quad u_{tt} = XT''$

$$X''T = XT'' \Rightarrow \frac{X''}{X} = \frac{T''}{T} = -\lambda$$

$X'' + \lambda X = 0$

$\lambda \le 0$ trivial solution only

$\lambda > 0$ let $\lambda = \mu^2$

$r^2 + \mu^2 = 0$

$r = \pm \mu i$

$X = A\cos\mu x + B\sin\mu x$

$0 = A(1) + B(0) \Rightarrow A = 0$

$X = B\sin\mu x$

$0 = B\sin\mu 9$

$\mu 9 = n\pi$

$\mu = \frac{n\pi}{9}$

$\Rightarrow \lambda = \frac{n^2\pi^2}{81}$

$T'' + \lambda T = 0$

$T'' + \frac{n^2\pi^2}{81}T = 0$

$r^2 + \frac{n^2\pi^2}{81} = 0$

$r = \pm\frac{n\pi}{9}i$

$T = C\cos\frac{n\pi}{9}t + D\sin\frac{n\pi}{9}t$

$T' = -\frac{n\pi}{9}C\sin\frac{n\pi}{9}t + \frac{n\pi}{9}D\cos\frac{n\pi}{9}t$

$0 = -\frac{n\pi}{9}C(0) + \frac{n\pi}{9}D(1)$

$\Rightarrow D = 0$

$T = C\cos\frac{n\pi}{9}t$

$u_n(x,t) = C_n \sin\frac{n\pi}{9}x \cos\frac{n\pi}{9}t$

$$C_n = \frac{2}{9}\int_0^9 u(x,0)\sin\frac{n\pi}{9}x\,dx = \frac{2}{9}\int_3^6 (x-1)\sin\frac{n\pi}{9}x\,dx$$

$u = x-1 \quad dv = \sin\frac{n\pi}{9}x$

$du = dx \quad v = \frac{-9}{n\pi}\cos\frac{n\pi}{9}x$

$$\frac{2}{9}\left[-(x-1)\frac{9}{n\pi}\cos\frac{n\pi}{9}x + \int_3^6 \frac{9}{n\pi}\cos\frac{n\pi}{9}x\,dx\right] =$$

$$\frac{2}{n\pi}(1-x)\cos\frac{n\pi}{9}x + \frac{81}{n^2\pi^2}\sin\frac{n\pi}{9}x\Big|_3^6 = \frac{2}{n\pi}\underset{-5}{(1-6)}\cos n\left(\tfrac{6}{9}\right)\pi + \frac{81}{n^2\pi^2}\sin\frac{n6\pi}{9} -$$

$$\frac{2}{n\pi}\underset{-2}{(1-3)}\cos n\left(\tfrac{3}{9}\right)\pi + \frac{81}{n^2\pi^2}\sin\frac{n3\pi}{9}$$

$$-\frac{10}{n\pi}\cos\frac{2n\pi}{3} + \frac{4}{n\pi}\cos\frac{n\pi}{3} + \frac{81}{n^2\pi^2}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right)$$

coeff does not simplify nicely

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{9}x\cos\frac{n\pi}{9}t$$

$$-\frac{10}{n\pi}\cos\frac{2n\pi}{3} + \frac{4}{n\pi}\cos\frac{n\pi}{3} + \frac{81}{n^2\pi^2}\left(\sin\frac{2n\pi}{3} - \sin\frac{n\pi}{3}\right)$$

$n=1$ $$\frac{-10}{\pi}\left(-\frac{1}{2}\right) + \frac{4}{\pi}\left(\frac{1}{2}\right) + \frac{81}{\pi^2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{7}{\pi}$$

$n=2$ $$\frac{-10}{2\pi}\left(-\frac{1}{2}\right) + \frac{4}{2\pi}\left(-\frac{1}{2}\right) + \frac{81}{4\pi^2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{3}{2\pi} - \frac{81\sqrt{3}}{4\pi^2}$$

$-\sqrt{3}$

$n=3$ $$\frac{-10}{3\pi}(1) + \frac{4}{3\pi}(-1) + \frac{81}{9\pi^2}(0-0) = -\frac{14}{3\pi}$$

$n=4$ $$\frac{-10}{4\pi}\left(-\frac{1}{2}\right) + \frac{4}{4\pi}\left(-\frac{1}{2}\right) + \frac{81}{16\pi^2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{81\sqrt{3}}{16\pi^2} + \frac{3}{4\pi}$$

$n=5$ $$\frac{-10}{5\pi}\left(-\frac{1}{2}\right) + \frac{4}{5\pi}\left(\frac{1}{2}\right) + \frac{81}{25\pi^2}\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{7}{5\pi}$$

$n=6$ $$-\frac{10}{6\pi}(1) + \frac{4}{6\pi}(1) + \frac{81}{36\pi^2}(0-0) = -\pi$$

10. The half-life of thorium-234 is 24.1 days. Set up a differential equation and solve it to model the loss of thorium-234 in a sample over time. If there were 1300μg of thorium-234 in a sample at the beginning, and only 18μg remain now, how old is the sample? Carry at least 4 significant digits throughout the calculation. (15 points)

$y' = ky \Rightarrow y = Ae^{kt}$

$.5 = e^{24.1k} \Rightarrow \ln\frac{1}{2} = 24.1k \Rightarrow k = -.02876129938$

$y = 1300e^{-.02876t}$

$\frac{18}{1300} = \frac{1300}{1300}e^{-.02876t}$

$\frac{\ln\left(\frac{18}{1300}\right)}{.02876} = .02876t$

$t = 148.8$ days

11. A spring-mass system has a spring constant of 9 N/m. A mass of 6 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance numerically equal to twice the magnitude of the instantaneous velocity. Suppose further that a driving force equal to cos(t) is added to the system. (35 points)

a. If the mass is set in motion from its equilibrium position with an initial velocity of 30 cm/sec, formulate the initial value problem describing the motion of the mass.

$6y'' + 2y' + 9y = \cos(t)$

$y(0) = 0$

$y'(0) = .3$ m/sec

b. Solve the equation you found in part a.

$$r = \frac{-2 \pm \sqrt{4-4(6)(9)}}{12} = \frac{-2 \pm \sqrt{212}\,i}{12} = \frac{-2 \pm 2\sqrt{53}\,i}{12} = -\frac{1}{6} \pm \frac{\sqrt{53}}{6}$$

$$y = Ae^{-1/6}\cos\frac{\sqrt{53}}{6}t + Be^{-1/6}\sin\frac{\sqrt{53}}{6}t + \frac{3}{13}\cos t + \frac{3}{13}\sin t$$

$Y(t) = C\cos t + D\sin t$

$Y'(t) = -C\sin t + D\cos t$

$Y''(t) = -C\cos t - D\sin t$

$-6C\cos t + 2D\cos t + 9C\cos t = \cos t$

$3C + 2D = 1$

$-6D\sin t - 2C\sin t + 9D\sin t = 0$

$3D - 2C = 0$

$C = 3/13 \quad D = 2/13$

$0 = A + 0 + \frac{3}{13} + 0 \qquad A = -\frac{3}{13}$

$y' = -\frac{1}{6}Ae^{-1/6}\cos\frac{\sqrt{53}}{6}t - \frac{\sqrt{53}}{6}Ae^{-1/6}\sin\frac{\sqrt{53}}{6}t - \frac{1}{6}Be^{-1/6}\sin\frac{\sqrt{53}}{6}t + \frac{\sqrt{53}}{6}Be^{-1/6}\cos\frac{\sqrt{53}}{6}t - \frac{3}{13}\sin t + \frac{2}{13}\cos t$

$3 = -\frac{1}{6}A + 0 + 0 + \frac{\sqrt{53}}{6}B + 0 + \frac{2}{13} \Rightarrow \frac{7}{65} = \frac{\sqrt{53}}{6}B \Rightarrow B = \frac{42}{65\sqrt{53}}$

$$\boxed{y = -\frac{3}{13}e^{-1/6}\cos\frac{\sqrt{53}}{6}t + \frac{42}{65\sqrt{53}}e^{-1/6}\sin\frac{\sqrt{53}}{6}t + \frac{3}{13}\cos t + \frac{2}{13}\sin t}$$

c. Describe the damping of the system: undamped, underdamped, critically damped or overdamped.

underdamped

d. What is the transient solution and what is the steady state solution, if they exist?

transient: $-\frac{3}{13}e^{-1/6}\cos\frac{\sqrt{53}}{6}t + \frac{42}{65\sqrt{53}}e^{-1/6}\sin\frac{\sqrt{53}}{6}t$

Steady state: $\frac{3}{13}\cos t + \frac{2}{13}\sin t$

e. Does the solution achieve resonance or does it contain beats? (Or neither.)

neither

though it appears to contain pseudo-"beats" until transient dies off

f. How does the natural frequency (without damping) compare to the quasi-frequency (if one exists)? What is the ratio of the two?

$6y'' + 9y = 0$

$y'' + \frac{3}{2}y = 0$

$\sqrt{r^2} = \sqrt{-\frac{3}{2}}$ $r = \pm\sqrt{\frac{3}{2}}\,i$ natural freq

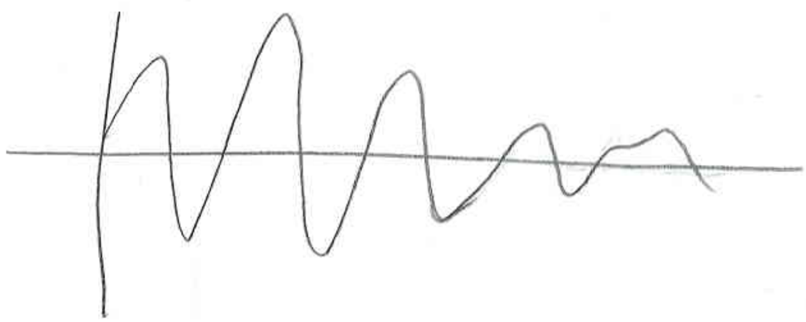
quasi-freq. $\frac{\sqrt{53}}{6}$

$\frac{\text{natural}}{\text{quasi}} = 1.0093...$

$\frac{\text{quasi}}{\text{natural}} = .99069...$

g. What is the long-term behaviour of the system as t→∞? Sketch the graph of the system for 0<t<10. Choose appropriate dimensions for y so that behaviour of the system can be seen clearly.

System will oscillate according to the forcing term



h. State the values of the first 4 times y=0 (if fewer than that, state them all for t>0). You should do this numerically.

$t = 2.81$

$t = 5.71$

$t = 8.59..$

$t = 11.47...$