Math 2255 Homework #1 Key

1 a. See attached page for graph $y' = 2y - 3 \quad (0,1)$

equation is autonomous

equilibrium solutions $0 = 2y - 3 \Rightarrow 3 = 2y \Rightarrow y = \frac{3}{2}$

$y > 3/2 \quad 2(5/2) - 3 = 5 - 3 = 2$

$y < 3/2 \quad 2(1) - 3 = 2 - 3 = -1$

unstable $y = 3/2$

threshold

as $t \to \infty$, from $(0,1)$ $y \to -\infty$ See also graph

b. see attached page for graph $y' = -1 - 2y \quad (0,0)$

autonomous

equilibrium solutions $0 = -1 - 2y \Rightarrow 1 = -2y \quad y = -1/2$

$y > -\frac{1}{2} \quad -1 - 2(0) = -1$

$y < -\frac{1}{2} \quad -1 - 2(-2) = 3$

Stable $y = -1/2$

N/A

as $t \to \infty$ from $(0,0)$ $y \to -1/2$ see also graph

c. $y' = y(4-y) \quad (1,2), (-1,5)$

See attached for graph

equation is autonomous

equilibrium solutions $0 = y(4-y) \Rightarrow y = 0, y = 4$

$y > 4 \quad 5(4-5) = -5$

$0 < y < 4 \quad 3(4-3) = 3$

$y < 0 \quad -1(4-(-1)) = -5$

$y = 0$ unstable N/A

$y = 4$ Stable, carrying capacity

as $t \to \infty$ from $(1,2)$ $y \to 4$ See also graph

as $t \to \infty$ from $(-1,5)$ $y \to 4$

d. $y' = y(y-2)^2$ See attached graphs

autonomous

equilibrium solutions $0 = y(y-2)^2$ $y=0, y=2$

$y>2$ $3(3-2)^2=3$

$0<y<2$ $1(3-1)^2=4$

$y<0$ $-1(3-(-1))^2=-16$

$y=2$ semi-stable neither

$y=0$ unstable N/A

as $t \to \infty$, from $(1,1)$ $y \to 2$

as $t \to \infty$ from $(-2,3)$ $y \to \infty$

e. $y' = y^2(y^2-1)$ $(-2,-2), (0,2)$ See attached graphs

autonomous

equilibrium solutions $0 = y^2(y^2-1) = y^2(y-1)(y+1)$

$y=0, y=1, y=-1$

$y>1$ $2^2(2-1)(2+1) = 12$

$0<y<1$ $\left(\frac{1}{2}\right)^2(1/2-1)(1/2+1) = -\frac{3}{16}$

$-1<y<0$ $\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}+1\right) = -\frac{3}{16}$

$y<-1$ $(-2)^2(-2-1)(-2+1) = -12$

$y=1$ threshold, unstable

$y=0$ semi-stable N/A

$y=-1$ stable N/A

as $t \to \infty$ from $(-2,-2)$ $y \to -1$

as $t \to \infty$ from $(0,2)$ $y \to \infty$

f. $y' = -2 + t - y$ $(0,0)$

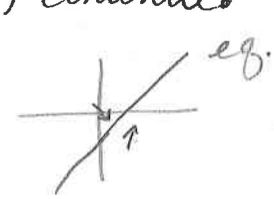
non-autonomous

equilibrium solutions $0 = -2 + t - y \Rightarrow y = -2 + t$

1f continued

checks points on either side of line

Slope at $(0,0) \Rightarrow \frac{dy}{dt} = -2$



as $t \to \infty$ from $(0,0) \Rightarrow$ line $y = -2+t \Rightarrow \infty$.

Slope at, say $(2,-4)$ $\frac{dy}{dt} = -2+2-(-4) = 4$

equilibrium appears to be stable (N/A)

(Check complete graph to be sure)

g. $y' = e^{-t} + y$ $(0,0)$ See attached for graph

nonautonomous

equilibrium solution $0 = e^{-t} + y \Rightarrow y = -e^{-t}$

at $(0,0)$ $\frac{dy}{dt} = e^0 + 0 = 1$

at $(0,-2)$ $\frac{dy}{dt} = e^0 - 2 = -1$ $\Rightarrow$ unstable, N/A

as $t \to \infty$ $y > -e^{-t}$, $y \to \infty$ (from $(0,0)$)

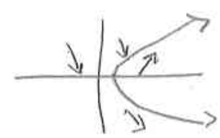
at $t \to \infty$, $y < -e^{-t}$, $y \to -\infty$

h. $y' = 2t - 1 - y^2$ $(-1,1)$ See attached for graph

non-autonomous

equilibrium solutions $0 = 2t-1-y^2 \Rightarrow y^2 = \underset{2(t-\frac{1}{2})}{2t-1} \Rightarrow y = \pm\sqrt{2t-1}$

at $(-1,1)$ $\frac{dy}{dt} = 2(-1) - 1 - 1^2 = -2-2 = -4$



at $(1,2)$ $\frac{dy}{dt} = 2(1) - 1 - 2^2 = 2-1-4 = 2-5 = -3$

at $(2,0)$ $\frac{dy}{dt} = 2(2) - 1 - 0 = 3$

at $(1,-2)$ $\frac{dy}{dt} = 2(1) - 1 - (-2)^2 = 2-1-4 = -3$

choose points in each region to determine stability

$y = +\sqrt{2t-1}$ stable (N/A)

$y = -\sqrt{2t-1}$ unstable (N/A)

at $t \to \infty$, $y \to -\infty$ from $(-1,1)$ Compare w/ graph

1i. $y' = e^y - 1 \quad (0,0)$

autonomous

equilibrium $\quad 0 = e^y - 1 \Rightarrow e^y = 1 \Rightarrow y = \ln 1 \Rightarrow y = 0$

from $(0,0)$ as $t \to \infty$, y stays 0, any other value of y: $y>0 \to \infty$, $y<0 \to -\infty$

$y=1 \quad \frac{dy}{dt} = e^1 - 1 > 0$

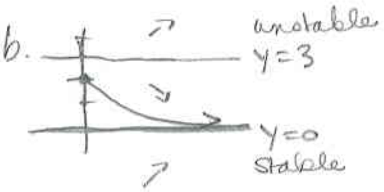
unstable (N/A)

$y=-1 \quad \frac{dy}{dt} = e^{-1} - 1 < 0$

2a.

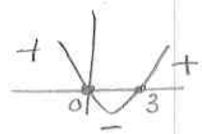
$\frac{dy}{dt} = y - 2$ or $k(y-2)$ $\quad k>0$ since $y=2$ is the equilibrium, and it is unstable

b.



$\frac{dy}{dt} = ky(y-3)$ since $y=0, y=3$ are equilibria

phase plane graph



$k>0$

3. $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 t)\, y = t^3$

this equation is a <u>linear</u>, 3rd order nonhomogeneous ordinary diff.-eq.

the $\frac{d^3y}{dt^3}$ is the third derivative; neither it nor $\frac{dy}{dt}$, nor y are multiplied by other y's or derivatives of y

the $\cos^2 t$ & t^3 don't make the equation non-linear since these don't contain y.

4.a. $y'' - y = 0 \qquad y_1(t) = e^t, \quad y_2(t) = \cosh(t)$

$y_1' = e^t, \; y_1'' = e^t \qquad e^t - e^t = 0$ ✓

$y_2' = \sinh t, \; y_2'' = \cosh t \qquad \cosh t - \cosh t = 0$

b. $ty' - y = t^2 \qquad y_1(t) = 3t + t^2 \qquad \cancel{3t} + 2t^2 - \cancel{3t} - t^2 = t^2$ ✓

$y_1'(t) = 3 + 2t,$

⑤

4c. $y^{IV} + 4y''' + 3y = t \quad y_1(t) = \frac{t}{3} \quad y_2(t) = \frac{t}{3} + e^{-t}$

$y_1' = \frac{1}{3} \quad y_1'' = 0 \quad y_1''' = 0 \quad y_1^{IV} = 0$

$y_2' = \frac{1}{3} - e^{-t} \quad y_2'' = e^{-t} \quad y_2''' = -e^{-t} \quad y_2^{IV} = e^{-t}$

$0 + 4(0) + 3(\frac{t}{3}) = t$ ✓

$e^{-t} + 4(-e^{-t}) + 3(\frac{t}{3} + e^{-t}) = \cancel{e^{-t}} \cancel{-4e^{-t}} + t + \cancel{3e^{-t}} = t$ ✓

d. $y'' + y = \sec(t) \quad y = \cos t \ln(\cos t) + t \sin t$

$y' = -\sin t (\ln \cos t) + \cancel{\cos t} \cdot \frac{1}{\cancel{\cos t}}(-\sin t) + \cancel{\sin t} + t\cos t = -\sin t \ln(\cos t) + t\cos t$

$y'' = -\cos t \ln \cos t - \sin t \frac{1}{\cos t}(-\sin t) + \cos t - t\sin t =$

$-\cos t \ln \cos t + \frac{\sin^2 t}{\cos t} + \cos t - t\sin t$

$\cancel{-\cos t \ln \cos t} + \frac{1-\cos^2 t}{\cos t} + \cos t \cancel{- t\sin t} + \cancel{\cos t \ln \cos t} + t \cancel{\sin t} =$

$\frac{1}{\cos t} - \cancel{\cos t} + \cancel{\cos t} = \sec t$ ✓

e. $y' - 2ty = 1 \quad y = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$

$y' = 2te^{t^2} \int_0^t e^{-s^2} ds + 2te^{t^2} + e^{t^2}(e^{-t^2}) = 2te^{t^2} \int_0^t e^{-s^2} ds + 2te^{t^2} + 1$

$2te^{t^2} \int_0^t e^{-s^2} ds + 2te^{t^2} + 1 - 2t(e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}) =$

$\cancel{2te^{t^2} \int_0^t e^{-s^2} ds} + \cancel{2te^{t^2}} + 1 - \cancel{2te^{t^2} \int_0^t e^{-s^2} ds} - \cancel{2te^{t^2}} = 1$ ✓

5a. $y'' + y' - 6y = 0 \quad y = e^{rt} \quad y' = re^{rt} \quad y'' = r^2 e^{rt}$

$r^2 e^{rt} + re^{rt} - 6e^{rt} = 0 \Rightarrow e^{rt}(r^2 + r - 6) = 0 \quad e^{rt} \neq 0$

$r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \quad r = -3, r = 2$

b. $t^2 y'' + 4ty' + 2y = 0 \quad y = t^r \quad y' = rt^{r-1} \quad y'' = r(r-1)t^{r-2} = (r^2 - r)t^{r-2}$

$t^2(r^2 - r)t^{r-2} + 4tr t^{r-1} + 2t^r = (r^2 - r)t^r + 4rt^r + 2t^r = t^r(r^2 - r + 4r + 2) = 0$

$t^r \neq 0$ unless $t = 0 \quad r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow r = 2, r = 1$

6a. $y' - 2y = 3e^t$ $\quad p(t) = -2 \quad \mu = e^{\int -2\,dt} = e^{-2t}$

$e^{-2t}y' - 2e^{-2t}y = 3e^t \cdot e^{-2t} = 3e^{-t}$

$\int (e^{-2t}y)' = \int 3e^{-t} \Rightarrow e^{-2t}y = -3e^{-t} + C$

$y = -3e^{-t} \cdot e^{2t} + Ce^{2t} = -3e^t + Ce^{2t}$

b. $ty' + 2y = \sin t \Rightarrow y' + \frac{2}{t}y = \frac{\sin t}{t}$ $\quad \mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

$t^2y' + 2ty = t\sin t \Rightarrow \int (t^2y)' = \int t \sin t$

$u = t \quad dv = \sin t\,dt$
$du = dt \quad v = -\cos t$

$t^2y = -t\cos t + \int t \cos t\,dt \Rightarrow$

$t^2y = -t\cos t + \sin t + C \Rightarrow y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$

c. $ty' - y = t^2e^{-t} \Rightarrow y' - \frac{1}{t}y = te^{-t}$ $\quad \mu = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}$

$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t} \Rightarrow \int (\frac{1}{t}y)' = \int e^{-t}\,dt$

$\frac{1}{t}y = -e^{-t} + C \Rightarrow y = -te^{-t} + Ct$

d. $y' - 2y = e^{2t}, \; y(0) = 2$ $\quad \mu = e^{\int -2\,dt} = e^{-2t}$

$e^{-2t}y' - 2e^{-2t}y = e^{-2t} \cdot e^{2t} = 1 \Rightarrow \int (e^{-2t}y)' = \int 1 \Rightarrow e^{-2t}y = t + c$

$y = te^{2t} + Ce^{2t}$ $\quad 2 = (0)e^0 + Ce^0 = C \Rightarrow C = 2$

$y(t) = te^{2t} + 2e^{2t}$

e. $t^3y' + 4t^2y = e^{-t} \quad y(-1) = 0 \Rightarrow y' + \frac{4}{t}y = t^{-3}e^{-t}$

$\mu = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4 \Rightarrow t^4y' + 4t^3y = te^{-t} \Rightarrow \int (t^4y)' = \int te^{-t}$

$u = t \quad dv = e^{-t}dt$
$du = dt \quad v = -e^{-t}$

$-te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + C = t^4y$

$y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$

be continued

$$0 = -\frac{e^1}{(-1)^3} - \frac{e^1}{(-1)^4} + \frac{C}{(-1)^4} \Rightarrow 0 = \cancel{e^1} - \cancel{e^1} + C \Rightarrow C = 0$$

$$y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4}$$

7a. $y' - 2y = 3e^t$ $\qquad \mu = e^{\int -2\,dt} = e^{-2t}$

$$y = e^{2t}\left[\int e^{-2t} \cdot 3e^t\,dt + C\right] = Ce^{2t} + \int 3e^{-t}\,dt = Ce^{2t} - 3e^{-t}$$

This agrees w/ 6a.

b. $ty' + 2y = \sin t \Rightarrow y' + \frac{2}{t}y = \frac{\sin t}{t}$ $\qquad \mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

$$y = \frac{1}{t^2}\left[\int t^2 \frac{\sin t}{t}\,dt + C\right] = \frac{C}{t^2} + \frac{1}{t^2}\int t \sin t\,dt$$

$u = t$ $\quad dv = \sin t$
$du = dt$ $\quad v = -\cos t$

$$= \frac{C}{t^2} - \frac{t\cos t}{t^2} + \frac{1}{t^2}\int \cos t\,dt = \frac{C}{t^2} - \frac{t\cos t}{t^2} + \frac{\sin t}{t^2} = \frac{C}{t^2} - \frac{\cos t}{t} + \frac{\sin t}{t^2}$$

agrees w/ 6b.

c. $ty' - y = t^2 e^{-t} \Rightarrow y' - \frac{1}{t}y = te^{-t}$ $\qquad \mu = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}$

$$y = t\left[\int \frac{1}{t} \cdot te^{-t}\,dt + C\right] = Ct + t\int e^{-t}\,dt = Ct - te^{-t}$$

agrees w/ 6c.

d. $y' - 2y = e^{2t}$ $\quad y(0) = 2$ $\qquad \mu = e^{\int -2\,dt} = e^{-2t}$

$$y = e^{2t}\left[\int e^{-2t} \cdot e^{2t}\,dt + C\right] = Ce^{2t} + e^{2t}\int 1\,dt = Ce^{2t} + te^{2t}$$

$2 = Ce^0 + 0e^0 = C$ $\qquad 2 = C$ $\qquad y = 2e^{2t} + te^{2t}$ $\quad$ agrees w/ 6d.

e. $t^3 y' + 4t^2 y = e^{-t}$ $\quad y(-1) = 0$ $\quad \Rightarrow y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$ $\qquad \mu = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$

$$y = \frac{1}{t^4}\left[\int t^4 \frac{e^{-t}}{t^3}\,dt + C\right] = \frac{C}{t^4} + \frac{1}{t^4}\int te^{-t}\,dt$$

$u = t$ $\quad dv = e^{-t}dt$
$du = dt$ $\quad v = -e^{-t}$

$$= \frac{C}{t^4} + \frac{1}{t^4}\left(-te^{-t} + \int e^{-t}\,dt\right) = \frac{C}{t^4} - \frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4}$$

$$0 = \frac{C}{(-1)^4} - \frac{e^1}{(-1)^3} - \frac{e^1}{(-1)^4} = \frac{C}{1} + \cancel{e^1} - \cancel{e^1} \Rightarrow C = 0 \qquad y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4}$$

agrees w/ 6e