Math 2255 Homework #3 Key

1. a. $y' = 2y - 1 \quad y(0) = 1 \quad h = .1$

$x_0 = 0 \quad y_0 = 1 \qquad m_0 = 2(1) - 1 = 1 \qquad m_0 = \frac{dy}{dt}(x_0, y_0)$

$x_1 = .1 \quad y_1 = 1.1 \qquad y_1 = 1(.1) + 1 = 1.1 \qquad y_1 = m_0(h) + y_0$

$m_1 = 2(1.1) - 1 = 1.2$

$y_2 = 1.2(.1) + 1 = .12 + 1.1 = 1.22$

$x_2 = .2 \quad y_2 = 1.22$

$m_2 = 2(1.22) - 1 = 2.42 - 1 = 1.44$

$y_3 = 1.44(.1) + 1.2 = 1.364$

$x_3 = .3 \quad y_3 = 1.364$

$m_3 = 2(1.364) - 1 = 1.728$

$y_4 = 1.728(.1) + 1.364 = 1.5368$

$x_4 = .4 \quad y_4 = 1.5368 \qquad m_4 = 2(1.5368) - 1 = 2.0736$

$y_5 = 2.0736(.1) + 1.5368 = 1.74416$

$\boxed{x_5 = .5 \quad y_5 = 1.7442}$

See attached Excel printouts for $h = .05$ & $h = .01$

b. $y' = y(3 - ty) \quad y(0) = 2 \quad h = .1$

$t_0 = 0 \quad y_0 = 2 \qquad m_0 = 2(3 - (0)(2)) = 6$

$y_1 = 6(.1) + 2 = 2.6$

$t_1 = .1 \quad y_0 = 2.6 \qquad m_0 = 2.6(3 - (.1)(2.6)) = 7.124$

$y_2 = 7.124(.1) + 2.6 = 3.3124$

$t_2 = .2 \quad y_2 = 3.3124 \qquad m_3 = 3.3124(3 - (.2)(3.3124)) = 7.742801248$

$y_3 = 7.7428\ldots(.1) + 3.3124 = 4.086680$

$t_3 = .3 \quad y_3 = 4.086680 \qquad m_4 = 4.086680(3 - (.3)(4.086680)) = 7.249754042$

$y_4 = 7.24975\ldots(.1) + 4.086680 = 4.811655404$

16 continued

$t_4 = .4 \quad y_4 = 4.811655404$

$m_0 = 4.811655(3-(.4)(4.811655)) = 5.174155121$

$y_5 = 5.174155(.1) + 4.811655 = 5.329070916$

$\boxed{t_5 = .5 \quad y_5 = 5.3291}$

for $h = .05$, & $h = .01$ see attached Excel printouts

1a. exact solution

$y' = 2(y - \frac{1}{2}) \Rightarrow \int \frac{dy}{y-\frac{1}{2}} = \int 2\,dt \Rightarrow \ln|y - \frac{1}{2}| = 2t + C$

$y - \frac{1}{2} = Ae^{2t} \Rightarrow y(t) = Ae^{2t} + \frac{1}{2}$

$y(0) = 1 \Rightarrow 1 = Ae^{0} + \frac{1}{2} \Rightarrow A = \frac{1}{2} \qquad y(t) = \frac{1}{2}e^{2t} + \frac{1}{2}$

$y(\frac{1}{2}) = \frac{1}{2}e^{(1)} + \frac{1}{2} \approx 1.859$ (compare w/ 1.845794 from Excel)

b. cannot be solved analytically

2a. $6r^2 - 5r + 1 = 0 \qquad (3r-1)(2r-1) = 0 \qquad r = \frac{1}{3} \quad r = \frac{1}{2}$

$y = c_1 e^{\frac{1}{3}t} + c_2 e^{\frac{1}{2}t} \qquad 4 = c_1 + c_2$

$y' = \frac{1}{3}c_1 e^{\frac{1}{3}t} + \frac{1}{2}c_2 e^{\frac{1}{2}t} \qquad 0 = \frac{1}{3}c_1 + \frac{1}{2}c_2$

$c_1 = 12 \quad c_2 = -8$

$y(t) = 12e^{\frac{1}{3}t} - 8e^{\frac{1}{2}t}$

$y'(t) = 4e^{\frac{1}{3}t} - 4e^{\frac{1}{2}t} = 0 \qquad \cancel{4}e^{\frac{1}{3}t} = \cancel{4}e^{\frac{1}{2}t}$

$\frac{1}{3}t = \frac{1}{2}t \Rightarrow t = 0$

critical point

$(0, 4)$

as $t \to -\infty$ approaches 0,

but $\to -\infty$ as $t \to \infty$

2b. $2r^2 - 3r + 1 = 0$ $\quad y(0) = 2,\ y'(0) = \frac{1}{2}$

$(2r-1)(r-1) = 0$ $\quad r = 1/2,\ r = 1$

$y = c_1 e^{1/2 t} + c_2 e^t$ $\quad 2 = c_1 + c_2$

$y' = 1/2 c_1 e^{1/2 t} + c_2 e^t$ $\quad \frac{1}{2} = \frac{1}{2} c_1 + c_2$

$c_1 = 3,\ c_2 = -1$

$y(t) = 3e^{1/2 t} - e^t$ $\quad y'(t) = \frac{3}{2} e^{1/2 t} - e^t = 0$

$\ln \frac{3}{2} e^{1/2 t} \overset{\ln}{=} e^t$

$\ln \frac{3}{2} + 1/2 t = t$

$\ln \frac{3}{2} = \frac{1}{2} t \Rightarrow t = 2 \ln(3/2)$

$\approx .8109$

approaches 0 as $t \to -\infty$

as $t \to \infty$, $y \to -\infty$

c. $r^2 + 4r + 5 = 0$ $\quad y(0) = 1,\ y'(0) = 0$

$r = \dfrac{-4 \pm \sqrt{16 - 20}}{2} \Rightarrow r = \dfrac{-4 \pm \sqrt{-4}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$ $\quad 1 = c_1$

$y'(t) = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t$

$0 = -2c_1 + c_2$

$0 = -2 + c_2 \Rightarrow c_2 = 2$

$y(t) = e^{-2t} \cos t + 2e^{-2t} \sin t$

many critical points, but as $t \to \infty$, y approaches 0.
oscillations are damped out.

d. $r^2 + 4r + 4 = 0$ $\quad y(-1) = 2,\ y'(-1) = 1$

$(r+2)^2 = 0$ $\quad r = -2$ repeated

$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

2d cont'd

$$2 = c_1 e^2 - c_2 e^2 \Rightarrow 2e^{-2} = c_1 - c_2$$

$$y'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$1 = -2c_1 e^2 + c_2 e^2 + 2c_2 e^2 = -2c_1 e^2 + 3c_2 e^2$$

$$e^{-2} = -2c_1 + 3c_2$$
$$2e^{-2} = c_1 - c_2 \;)3$$

$$e^{-2} = -2c_1 + \cancel{3c_2}$$
$$6e^{-2} = 3c_1 - \cancel{3c_2}$$
$$7e^{-2} = c_1$$

$$2e^{-2} = 7e^{-2} - c_2$$
$$c_2 = 5e^{-2}$$

$$y(t) = 7e^{-2t-2} + 5te^{-2t-2} \Rightarrow 7e^{-2(t-1)} + 5te^{-2(t-1)}$$

3a. $r^2 + 2 = 0$ $\quad r = \pm\sqrt{2}i$ $\quad y(t) = c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t$

$y'(0) = 0,\ y'(\pi) = 0$

$$y'(t) = -\sqrt{2}c_1 \sin\sqrt{2}t + \sqrt{2}c_2 \cos\sqrt{2}t$$

$$0 = 0 + \sqrt{2}c_2(1) \Rightarrow c_2 = 0$$

$$0 = -\sqrt{2}c_1 \sin(\pi) = 0 \Rightarrow c_1 = \text{anything}$$

$y(t) = c_1 \cos\sqrt{2}t$ (infinite #'s of solutions)

b. $y'' + y = 0 \quad y(0) = 0,\ y(L) = 0$

$r^2 + 1 = 0 \quad r = \pm i \quad y(t) = c_1 \cos t + c_2 \sin(t)$

$$0 = c_1(1) + c_2(0) \Rightarrow c_1 = 0$$

$0 = c_2 \sin(L) = 0 \Rightarrow c_2 = 0$ unless $L = n\pi$

for $L = n\pi$ $\quad y(t) = c_2 \sin t \Rightarrow$ infinite #'s of solutions

for all other L $\quad c_2 = 0 \Rightarrow y(t) = 0$ identically

3c. $x^2y'' + 3xy' + y = x^2 \quad y(1)=0 \quad y(e)=0$

(5)

$y = x^n \quad y' = nx^{n-1}, \quad y'' = (n^2-n)x^{n-2}$

$(n^2-n)x^2 + 3nx^n + x^n = 0$

$n^2 - n + 3n + 1 = 0 \Rightarrow n^2 + 2n + 1 = 0 \Rightarrow (n+1)^2 = 0$

$y_c(x) = \frac{c_1}{x} + \frac{c_2 \ln x}{x} \qquad y = x^{-1}, \ \ln x \cdot x^{-1}$

$Ax^2 + Bx + C = y_p$

$y_p' = 2Ax + B \quad y_p'' = 2A$

$2Ax^2 + 3x(2Ax+B) + Ax^2 + Bx + C = x^2$

$2Ax^2 + 6Ax^2 + 3Bx + Ax^2 + Bx + \cancel{C} = x^2 \qquad C=0$

$4Bx = 0 \Rightarrow B = 0$

$(2A + 6A + A)x^2 = (1)x^2 \Rightarrow 9A = 1 \quad A = 1/9 \qquad y_p = 1/9\, x^2$

$y(x) = \frac{c_1}{x} + \frac{c_2 \ln x}{x} + \frac{1}{9}x^2$

$0 = \frac{c_1}{1} + \frac{c_2(0)}{1} + \frac{1}{9}(1)^2 \Rightarrow 0 = c_1 + \frac{1}{9} \Rightarrow c_1 = \frac{-1}{9}$

$0 = \frac{-1}{9e} + \frac{c_2 \ln e}{e} + \frac{1}{9}e^2 = \frac{-1}{9e} + \frac{c_2}{e}\frac{9}{9} + \frac{1}{9}e^2 \cdot \frac{e}{e} \Rightarrow \frac{-1 + 9c_2 + e^3}{9e}$

$-1 + 9c_2 + e^3 = 0 \qquad \cancel{9}c_2 = \frac{1-e^3}{9}$

$y(x) = -\frac{1}{9x} + \frac{(1-e^3)\ln x}{9x} + \frac{1}{9}x^2$

$\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} : \cos^2 t + \sin^2 t = 1 \neq 0$ yes, fundamental set

4b. $\begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = \cancel{xe^x} + x^2e^x - \cancel{xe^x} = x^2e^x \neq 0$ unless $x=0$ 6

fundamental set

c. $\begin{vmatrix} e^t\sin t & e^t\cos t \\ e^t\sin t + e^t\cos t & e^t\cos t - e^t\sin t \end{vmatrix} = \cancel{e^{2t}\cos t\sin t} - e^{2t}\sin^2 t - \cancel{e^{2t}\sin t}\cos t - e^{2t}\cos^2 t$

$= -e^{2t}(\sin^2 t + \cos^2 t) = -e^{2t} \neq 0$

fundamental set

d. $\begin{vmatrix} t^2 & t^2\ln t & t^{-4} \\ 2t & 2t\ln t + t & -4t^{-5} \\ 2 & \underset{+3}{2\ln t + 2 + 1} & 20t^{-6} \end{vmatrix} = t^2\left[(2t\ln t + t)(20t^{-6}) + 4t^{-5}(2\ln t + 3)\right]$

$- t^2\ln t\left[2t \cdot 20t^{-6} + 8t^{-5}\right] +$

$t^{-4}\left[2t(2\ln t + 3) - 2(2t\ln t + t)\right] =$

$\cancel{40t^{-3}\ln t} + 20t^{-3} + \cancel{8t^{-3}\ln t} + 12t^{-3} - \cancel{40t^{-3}\ln t} - \cancel{8t^{-3}\ln t} +$

$\cancel{4t^{-3}\ln t} + 6t^{-3} - \cancel{4t^{-3}\ln t} + 2t^{-3} = 40t^{-3} \neq 0$

fundamental set

e. $\begin{vmatrix} \sinh t & \cosh t & e^t \\ \cosh t & \sinh t & e^t \\ \sinh t & \cosh t & e^t \end{vmatrix} = \sinh t\,(e^t\sinh t - e^t\cosh t) -$

$\cosh t\,(e^t\cosh t - e^t\sinh t) +$

$e^t\underbrace{(\cosh^2 t - \sinh^2 t)}_{=1} =$

$e^t\sinh^2 t - \cancel{e^t\cosh t\sinh t} - e^t\cosh^2 t + \cancel{e^t\sinh t\cosh t} + e^t =$

$e^t(-\cosh^2 t + \sinh^2 t) + e^t = -e^t + e^t = 0$ not a fundamental

5a. $ty'' + 3y = t \qquad y(1) = 1, \; y'(1) = 2$

$W(t) = ce^{-\int p(t)\,dt} \qquad y'' + \frac{3}{t}y = 1$

$y'' + p(t)y' + q(t)y = 0$

$p(t) = 0$ here $\quad W(t) = ce^{-\int 0\,dt} = ce^{0} = c \neq 0$

defined for all t

6. $t(t-4)y'' - 3ty' + 4y = 2 \qquad y(3) = 0, \; y'(3) = -1$

$y'' + \frac{-3}{(t-4)}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$

$p(t) = \frac{-3}{(t-4)}$

$W(t) = e^{+\int \frac{+3}{t-4}dt} = e^{3\ln|t-4|} = e^{\ln(t-4)^3} = (t-4)^3$

$\neq 0$ unless $t = 4$

$(-\infty, 4) \cup (4, \infty)$ for these initial conditions use $(-\infty, 4)$

7. $(1 - x\cot x)y'' - xy' + y = 0 \qquad y(\pi/2) = 1 \quad y'(\pi/2) = 0$

$y'' - \frac{x}{1 - x\cot x}y' + \frac{1}{1 - x\cot x}y = 0$

$p(t) = \frac{x}{1 - x\cot x} = \frac{x\sin x}{1 - x\cos x}$

$W(t) = ce^{\int \frac{x}{1 - x\cot x}dx} \neq 0$ but $p(t)$ will be undefined

when $x =$ multiples of π (since $\cot x$ is undefined there)

... $1 - x\cos x = 0$ (≈ 4.9 rad, 7.72 rad, etc)

6a. $e^{1+2i} = e^1 e^{2i} = e^1(\cos 2 + i \sin 2) = e\cos(2) + ei\sin(2)$ ⑧

b. $2^{1-i} = e^{(\ln 2)(1-i)} = e^{\ln 2 - (\ln 2)i} = e^{\ln 2}(\cos(\ln 2) + i\sin(\ln 2)) =$
$2\cos(\ln 2) + 2i\sin(\ln 2)$

c. $e^{2-\pi/2 i} = e^2(\cos(-\pi/2) + i\sin(-\pi/2)) = e^2(0 - i\sin \pi/2) = e^2(-i(1)) =$
$-e^2 i$

7a. $x^2y'' + xy' + y = 0$ let $y = x^n$ $y' = nx^{n-1}$ $y'' = n(n-1)x^{n-2}$

$(n^2-n)x^n + nx^n + x^n = 0$ $x^n[n^2 \cancel{-n} + \cancel{n} + 1] = 0$ $n^2+1=0$

$n = \pm i$

$y = x^n = x^i, x^{-i} \Rightarrow e^{(\ln x)i} = \cos(\ln x) + i\sin(\ln x)$

$y(x) = c_1\cos(\ln x) + c_2\sin(\ln x)$

b. $t^2y'' - ty' + 5y = 0$

$n^2 - n - n + 5 = 0 \Rightarrow n^2 - 2n + 5 = 0$

$$n = \frac{+2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$y = t^n = t^{1+2i} = t(t^{2i}) = te^{(\ln t)2i} = te^{(\ln t^2)i} = t(\cos(\ln t^2) + i\sin(\ln t^2))$

$y(t) = c_1 t\cos(\ln t^2) + c_2 t\sin(\ln t^2)$

. $t^2y'' + 5ty' + 13y = 0$

$n^2 - n + 5n + 13 = 0 \Rightarrow n^2 + 4n + 13 = 0$

$$n = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

1a.

N	Xn	Yn	Mn		
0	0	1	1	H=	0.1
1	0.1	1.1	1.2		
2	0.2	1.22	1.44		
3	0.3	1.364	1.728		
4	0.4	1.5368	2.0736		
5	0.5	1.74416			
6					

N	Xn	Yn	Mn		
0	0	1	1	H=	0.05
1	0.05	1.05	1.1		
2	0.1	1.105	1.21		
3	0.15	1.1655	1.331		
4	0.2	1.23205	1.4641		
5	0.25	1.305255	1.61051		
6	0.3	1.385781	1.771561		
7	0.35	1.474359	1.948717		
8	0.4	1.571794	2.143589		
9	0.45	1.678974	2.357948		
10	0.5	1.796871			
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1a

N	Xn	Yn	Mn		
0	0	1	1	H=	0.01
1	0.01	1.01	1.02		
2	0.02	1.0202	1.0404		
3	0.03	1.030604	1.061208		
4	0.04	1.041216	1.082432		
5	0.05	1.05204	1.104081		
6	0.06	1.063081	1.126162		
7	0.07	1.074343	1.148686		
8	0.08	1.08583	1.171659		
9	0.09	1.097546	1.195093		
10	0.1	1.109497	1.218994		
11	0.11	1.121687	1.243374		
12	0.12	1.134121	1.268242		
13	0.13	1.146803	1.293607		
14	0.14	1.159739	1.319479		
15	0.15	1.172934	1.345868		
16	0.16	1.186393	1.372786		
17	0.17	1.200121	1.400241		
18	0.18	1.214123	1.428246		
19	0.19	1.228406	1.456811		
20	0.2	1.242974	1.485947		
21	0.21	1.257833	1.515666		
22	0.22	1.27299	1.54598		
23	0.23	1.28845	1.576899		
24	0.24	1.304219	1.608437		
25	0.25	1.320303	1.640606		
26	0.26	1.336709	1.673418		
27	0.27	1.353443	1.706886		
28	0.28	1.370512	1.741024		
29	0.29	1.387922	1.775845		
30	0.3	1.405681	1.811362		
31	0.31	1.423794	1.847589		
32	0.32	1.44227	1.884541		
33	0.33	1.461116	1.922231		
34	0.34	1.480338	1.960676		
35	0.35	1.499945	1.99989		
36	0.36	1.519944	2.039887		
37	0.37	1.540343	2.080685		
38	0.38	1.561149	2.122299		
39	0.39	1.582372	2.164745		
40	0.4	1.60402	2.20804		
41	0.41	1.6261	2.2522		
42	0.42	1.648622	2.297244		
43	0.43	1.671595	2.343189		
44	0.44	1.695027	2.390053		
45	0.45	1.718927	2.437854		

46	0.46	1.743306	2.486611
47	0.47	1.768172	2.536344
48	0.48	1.793535	2.58707
49	0.49	1.819406	2.638812
50	0.5	1.845794	

1b.

N	Xn	Yn	Mn		
0	0	2	6	H=	0.1
1	0.1	2.6	7.124		
2	0.2	3.3124	7.742801		
3	0.3	4.08668	7.249754		
4	0.4	4.811656	5.174155		
5	0.5	5.329071			

N	Xn	Yn	Mn		
0	0	2	6	H=	0.05
1	0.05	2.3	6.6355		
2	0.1	2.631775	7.202701		
3	0.15	2.99191	7.633001		
4	0.2	3.37356	7.844499		
5	0.25	3.765785	7.752071		
6	0.3	4.153389	7.284975		
7	0.35	4.517637	6.409746		
8	0.4	4.838125	5.151394		
9	0.45	5.095694	3.602338		
10	0.5	5.275811			

1b

N	Xn	Yn	Mn		
0	0	2	6	H=	0.01
1	0.01	2.06	6.137564		
2	0.02	2.121376	6.274122		
3	0.03	2.184117	6.40924		
4	0.04	2.248209	6.54245		
5	0.05	2.313634	6.673256		
6	0.06	2.380366	6.80113		
7	0.07	2.448378	6.925514		
8	0.08	2.517633	7.04582		
9	0.09	2.588091	7.161434		
10	0.1	2.659705	7.271713		
11	0.11	2.732422	7.375993		
12	0.12	2.806182	7.473588		
13	0.13	2.880918	7.563795		
14	0.14	2.956556	7.645897		
15	0.15	3.033015	7.719168		
16	0.16	3.110207	7.782879		
17	0.17	3.188036	7.8363		
18	0.18	3.266399	7.878711		
19	0.19	3.345186	7.909406		
20	0.2	3.42428	7.927701		
21	0.21	3.503557	7.932939		
22	0.22	3.582886	7.924502		
23	0.23	3.662131	7.901817		
24	0.24	3.741149	7.86436		
25	0.25	3.819793	7.811674		
26	0.26	3.89791	7.743367		
27	0.27	3.975343	7.659124		
28	0.28	4.051935	7.558715		
29	0.29	4.127522	7.441999		
30	0.3	4.201942	7.308931		
31	0.31	4.275031	7.159567		
32	0.32	4.346627	6.994068		
33	0.33	4.416567	6.8127		
34	0.34	4.484694	6.615839		
35	0.35	4.550853	6.403967		
36	0.36	4.614893	6.177674		
37	0.37	4.676669	5.937651		
38	0.38	4.736046	5.684688		
39	0.39	4.792893	5.419668		
40	0.4	4.847089	5.143558		
41	0.41	4.898525	4.857401		
42	0.42	4.947099	4.562306		
43	0.43	4.992722	4.259439		
44	0.44	5.035316	3.950008		
45	0.45	5.074816	3.635256		

46	0.46	5.111169	3.316445
47	0.47	5.144333	2.994842
48	0.48	5.174282	2.671713
49	0.49	5.200999	2.348306
50	0.5	5.224482	