Differential Equations Math 2255 Homework #4 Key

1. a. $t^2y'' + 2ty' - 2y = 0 \quad t>0, \quad y_1(t) = t$

$y(t) = v(t)t \qquad y'(t) = v'(t)t + v(t) \qquad y''(t) = v''(t)t + 2v'(t)$

$t^3v''(t) + 2t^2v'(t) + 2t^2v'(t) + \cancel{2tv(t)} - \cancel{2tv(t)} = 0$

$$\frac{t^3v''(t) + 4t^2v'(t) = 0}{t^2} \Rightarrow tv''(t) + 4v'(t) = 0$$

let $u = v'(t) \Rightarrow tu' + 4u = 0 \Rightarrow \frac{tu'}{u} = -\frac{4u}{u}$

$$\int \frac{du}{u} = \int -\frac{4}{t}\,dt \Rightarrow \ln u = -4\ln t + c \Rightarrow \cancel{\ln} u = \cancel{\ln} t^{-4}(c_1)$$

$$u = t^{-4} \Rightarrow v'(t) = t^{-4} \Rightarrow v = \int t^{-4}\,dt = -\tfrac{1}{3}t^{-3} + C_2$$

$$\Rightarrow y(t) = C_1(t^{-3})(t) + C_2t \Rightarrow C_1t^{-2} + C_2t$$

b. $(x-1)y'' - xy' + y = 0 \quad x>1 \quad y_1(x) = e^x$

$y(x) = v(x)e^x \qquad y'(x) = v'(x)e^x + v(x)e^x \qquad y''(x) = v''(x)e^x + 2v'(x)e^x + v(x)e^x$

$$(x-1)[v''e^x + 2v'e^x + ve^x] - x[v'e^x + ve^x] + ve^x = 0$$

$$\cancel{e^x}[v''x + 2xv' + \cancel{vx} - v'' - 2v' - \cancel{v} - v'x - \cancel{vx} + \cancel{v}] = 0$$

$v''(x-1) + v'(x-2) = 0 \qquad$ let $u = v' \quad u' = v''$

$$\frac{u'(\cancel{x-1})}{u} = -\frac{\cancel{u}(x-2)}{x-1} \Rightarrow \int \frac{du}{u} = \int \frac{x-2}{x-1}\,dx = \int 1 - \frac{1}{x-1}\,dx$$

$$\begin{array}{r} 1 \\ x-1 \overline{)\, x-2} \\ -x+1 \\ \hline -1 \end{array}$$

$\ln u = x - \ln|x-1| = \ln e^x - \ln|x-1|$

$\cancel{\ln} u = \cancel{\ln}\left[\frac{e^x}{x-1}\right]$

$u = \frac{e^x}{x-1}$

1b. cont'd. ②

$V = \int \frac{e^x}{x-1}\,dx \qquad y(x) = c_1 e^x \int \frac{e^x}{x-1}\,dx + c_2 e^x$

this integral would have to be done as a series.

c. $xy'' - y' + 4x^3y = 0, \quad x>0, \quad y_1(x) = \sin x^2$

$y(x) = v(x)\sin x^2 \qquad y'(x) = v'\sin x^2 + v\cdot 2x\cos x^2$

$y''(x) = v''\sin x^2 + 4xv'\cos x^2 + 2v\cos x^2 - 4x^2 v\sin x^2$

$xv''\sin x^2 + 4x^2v'\cos x^2 + \cancel{2vx\cos x^2} - \cancel{4x^3v\sin x^2} - v'\sin x^2 - \cancel{2xv\cos x^2}$

$+ \cancel{4x^3v\sin x^2} = 0$

$v''x\sin x^2 + v'(4x^2\cos x^2 - \sin x^2) = 0 \qquad$ let $u = v' \quad u' = v''$

$\frac{v''x\sin x^2}{v'x\sin x^2} = \frac{v'(\sin x^2 - 4x^2\cos x^2)}{v'\,x\sin x^2} \Rightarrow \int \frac{du}{u} = \int \frac{1}{x} - 4x\cot x^2\,dx$

$\cancel{\ln} u = \ln x - 2\ln|\sin x^2| = \ln x - \ln\sin^2 x^2 = \cancel{\ln}|x\csc^2 x^2|$

$u = x\csc^2 x^2 \Rightarrow \quad V = \int x\csc^2 x^2\,dx = -\frac{1}{2}\cot x^2 + c_2$

$y(x) = c_1\cot x^2\cdot\sin x^2 + c_2\sin x^2$

$= c_1\cos x^2 + c_2\sin x^2$

2.a. $2y'' + 3y' + y = t^2 + 3\sin t \qquad y(0) = 0 \quad y'(0) = 1$

$2r^2 + 3r + 1 = 0$

$(2r+1)(r+1) = 0 \Rightarrow r = -1/2, \ r = -1 \qquad y_c(t) = c_1 e^{-1/2 t} + c_2 e^{-t}$

$Y(t) = At^2 + Bt + C + D\cos t + E\sin t$

$Y'(t) = 2At + B - D\sin t + E\cos t$

$Y''(t) = 2A - D\cos t - E\sin t$

$4A - 2D\cos t - 2E\sin t + 6At + 3B - 3D\sin t + 3E\cos t + At^2 + Bt + C$

$+ D\cos t + E\sin t = t^2 + 3\sin t$

$4A + 3B + C = 0 \qquad$ constants

$6A + B = 0 \qquad$ t's

2a cont'd

$A = 1$ $\quad t^2$'s

$-2D + 3E + D = 0$ $\quad$ cost's

$-2E \pm 3D + E = 3$ $\quad$ sint's

$\Downarrow$

$-D + 3E = 0 \Rightarrow D = 3E$

$-E - 3D = 3 \Rightarrow -E - 3(3E) = -10E = 3 \Rightarrow E = -3/10 \Rightarrow D = -9/10$

$A = 1 \Rightarrow 6A + B = 0 \Rightarrow B = -6A$

$\Rightarrow B = -6$

$\Rightarrow 4(1) + 3(-6) + C = 0 \Rightarrow$

$-14 + C = 0 \quad C = 14$

$y_p(t) = t^2 - 6t + 14 - 9/10 \cos t - 3/10 \sin t$

$y(t) = c_1 e^{-1/2 t} + c_2 e^{-t} + t^2 - 6t + 14 - 9/10 \cos t - 3/10 \sin t$

$0 = c_1 + c_2 + 0 - 0 + 14 - 9/10 - 0 \Rightarrow c_1 + c_2 = -\frac{131}{10}$

$y'(t) = -1/2 c_1 e^{-1/2 t} - c_2 e^{-t} + 2t - 6 + 9/10 \sin t - 3/10 \cos t$

$1 = -1/2 c_1 - c_2 + 0 - 6 + 0 - 3/10 \Rightarrow -1/2 c_1 - c_2 = 1 + \frac{63}{10} = \frac{73}{10}$

$$\begin{array}{l} c_1 + \cancel{c_2} = -\frac{131}{10} \\ -1/2 c_1 - \cancel{c_2} = \frac{73}{10} \\ \hline \end{array}$$

$\cancel{1/2} c_1 = \frac{-58}{\cancel{10}_5} \cdot \cancel{2} \Rightarrow c_1 = -\frac{58}{5}$

$c_2 = -\frac{131}{10} + \frac{58}{5} = -\frac{131}{10} + \frac{116}{10} = \frac{-15}{10} = -\frac{3}{2}$

$y(t) = -\frac{58}{5} e^{-1/2 t} - \frac{3}{2} e^{-t} + t^2 - 6t + 14 - 9/10 \cos t - 3/10 \sin t$

2b. $y'' + y' + 4y = 2 \sinh t \quad y(0) = 1, \; y'(0) = 0$

$r = \frac{-1 \pm \sqrt{1 - 16}}{2} = \frac{-1 \pm \sqrt{15} i}{2} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i \quad y(t) = c_1 e^{-1/2 t} \cos\left(\frac{\sqrt{15}}{2} t\right) + c_2 e^{-1/2 t} \sin\left(\frac{\sqrt{15}}{2} t\right)$

$Y(t) = A \cosh t + B \sinh t \quad Y'(t) = A \sinh t + B \cosh t$

$Y''(t) = A \cosh t + B \sinh t$

$A \cosh t + B \sinh t + A \sinh t + B \cosh t + 4A \cosh t + 4B \sinh t = 2 \sinh t$

$\cosh t: \; (A + B + 4A) = 0 \Rightarrow 5A + B = 0 \quad B = -5A$

$\sinh t: \; (B + A + 4B) = 2 \Rightarrow 5B + A = 2 \quad 5(-5A) + A = 2 \Rightarrow -24A = 2$

$-25A + A$

2b cont'd

$A = -1/12 \Rightarrow B = \frac{5}{12}$

$Y_p(t) = -1/12 \cosh t + \frac{5}{12} \sinh t$

$y(t) = c_1 e^{-1/2 t} \cos(\frac{\sqrt{15}}{2}t) + c_2 e^{-1/2 t} \sin(\frac{\sqrt{15}}{2}t) - 1/12 \cosh t + \frac{5}{12} \sinh t$

$1 = c_1 - 1/12 \Rightarrow c_1 = 13/12$

$y'(t) = -1/2 c_1 e^{-1/2 t} \cos(\frac{\sqrt{15}}{2}t) - \frac{\sqrt{15}}{2} c_2 e^{-1/2 t} \cancel{\sin}(\frac{\sqrt{15}}{2}t) - 1/2 c_2 e^{-1/2 t} \sin(\frac{\sqrt{15}}{2}t) +$
$\frac{\sqrt{15}}{2} c_2 e^{-1/2 t} \cos(\frac{\sqrt{15}}{2}t) - 1/12 \sinh t + 5/12 \cosh t$

$0 = -1/12(\frac{13}{2}) + \frac{\sqrt{15}}{2} c_2 + \frac{5}{12} \Rightarrow -\frac{13}{24} + \frac{10}{24} + \frac{\sqrt{15}}{2} c_2 = 0 \Rightarrow \frac{+\cancel{3}1}{\cancel{24}\,4} = \frac{\sqrt{15}}{2} c_2$

$\frac{1}{4} = \sqrt{15}\, c_2 \Rightarrow c_2 = \frac{1}{4\sqrt{15}}$

$y(t) = \frac{13}{12} e^{-1/2 t} \cos(\frac{\sqrt{15}}{2}t) + \frac{1}{4\sqrt{15}} e^{-1/2 t} \sin(\frac{\sqrt{15}}{2}t) - \frac{1}{12} \cosh t + \frac{5}{12} \sinh t$

3a. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$y_1 = e^{-t} \cos t$

$y_2 = e^{-t} \sin t$

$g(t) = 2e^{-t} \cos t + 4e^{-t} t^2 \sin t$

$Y(t) = t(At^2 + Bt + C) e^{-t} \cos t + t(Dt^2 + Et + F) e^{-t} \sin t$

b. $y'' + 4y = t^2 \sin 2t + (6t + 7) \cos 2t$

$r^2 + 4 = 0 \quad r = \pm 2i \qquad y_1 = \cos 2t,\ y_2 = \sin 2t$

$Y(t) = t(At^2 + Bt + C) \cos 2t + t(Dt^2 + Et + F) \sin 2t$

4.a. $y'' + y = \tan t$

$y_1 = \cos t,\ y_2 = \sin t$

$r^2 + 1 = 0 \qquad r = \pm i$

$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

4a cont'd.

$$y(t) = -\cos t \int \frac{\sin t \cdot \tan t}{1}\, dt + \sin t \int \frac{\cos t \tan t}{1}\, dt =$$

$$= -\cos t \int \frac{\sin^2 t}{\cos t}\, dt + \sin t \int \sin t\, dt =$$

$$= -\cos t \int \frac{1-\cos^2 t}{\cos t}\, dt + \sin t \int \sin t\, dt =$$

$$= -\cos t \int \sec t - \cos t\, dt + \sin t \int \sin t\, dt =$$

$$-\cos t\left[\ln|\sec t + \tan t| - \sin t + C_1\right] + \sin t\left[-\cos t + C_2\right]$$

$$\cos t \ln|\sec t + \tan t| + \sin t \cos t + c_1 \cos t - \sin t \cos t + C_2 \sin t$$

$$y(t) = c_1 \cos t + c_2 \sin t + \cos t \ln|\sec t + \tan t|$$

b. $y'' - 2y' + y = \frac{e^t}{1+t^2}$ $\quad y_1 = e^t \quad y_2 = te^t$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}$$

$$y(t) = -e^t \int \frac{e^t \cdot te^t}{(1+t^2)e^{2t}}\, dt + te^t \int \frac{e^t \cdot e^t}{e^{2t}(1+t^2)}\, dt$$

$$= -e^t \int \frac{t}{1+t^2}\, dt + te^t \int \frac{1}{1+t^2}\, dt =$$

$$-e^t\left(\tfrac{1}{2}\ln|1+t^2| + C_1\right) + te^t(\arctan t + C_2) =$$

$$C_1 e^t + C_2 te^t + \tfrac{1}{2}e^t \ln|1+t^2| + te^t \arctan t$$

c. $ty'' - (1+ty') + y = t^2 e^{2t}$ $\quad y_1 = 1+t, \quad y_2 = e^t$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = (1+t)e^t - e^t = te^t$$

$$y(t) = -(1+t)\int \frac{t^2 e^{2t}}{te^t} e^t\, dt + e^t \int \frac{(1+t)t^2 e^{2t}}{e^t}\, dt$$

$$= -(1+t)\int te^{2t}\, dt + e^t \int (t^2 + t^3)e^t\, dt$$

4c cont'd

u	dv
t	e^{2t}
1	$\frac{1}{2}e^{t}$
	$\frac{1}{4}e^{t}$

u	dv
t^2+t^3	e^t
$2t+3t^2$	e^t
$2+6t$	e^t
6	e^t
	e^t

$$y(t) = -(1+t)\left[\tfrac{1}{2}te^{2t} - \tfrac{1}{4}e^{2t} + c_1\right] + e^t\left[(t^2+t^3)e^t - (2t+3t^2)e^t + (2+6t)e^t - 6e^t + c_2\right]$$

$$c_1(1+t) + c_2e^t + e^{2t}\left[-\cancel{\tfrac{1}{2}t} + \tfrac{1}{4} - \cancel{\tfrac{1}{2}t^2} + \cancel{\tfrac{1}{4}t} + \cancel{t^2} + t^3 - 2t - 3t^2 + 2 + 6t - 6\right]$$

$$= c_1(1+t) + c_2e^t + e^{2t}\left[t^3 - \tfrac{5}{2}t^2 + \tfrac{7}{4}t - \tfrac{7}{4}\right]$$

4d. $x^2y'' - 3xy' + 4y = x^2$ $\qquad y_1 = x^2$, $y_2 = x^2\ln x$

$$y(x) = -x^2\int \frac{x^2 \cdot x^2\ln x}{x^3}\,dx + x^2\ln x\int \frac{x^2 \cdot x^2}{x^3}\,dx$$

$$W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = \frac{\cancel{2x^3\ln x} + x^3 - \cancel{2x^3\ln x}}{x^3} =$$

$$= -x^2\int x\ln x\,dx + x^2\ln x\int x\,dx$$

$$-x^2\left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx\right] + x^2\ln x\left[\tfrac{1}{2}x^2 + c_2\right]$$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}\,dx \quad v = \frac{1}{2}x^2$

$$= -x^2\left[\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c_1\right] + x^2\ln x\left[\tfrac{1}{2}x^2 + c_2\right]$$

$$= c_1x^2 + c_2x^2\ln x \cancel{-\tfrac{1}{2}x^4\ln x} + \cancel{\tfrac{1}{2}x^4\ln x} + \tfrac{1}{4}x^4 =$$

$$\boxed{c_1x^2 + c_2x^2\ln x + \tfrac{1}{4}x^2}$$

5. undetermined coefficients ok: e^t, $\cos t$, $\sinh t$, t^2, $e^t\cos t$, te^t, etc.

must use variation of parameters: $\ln t$, $\sec t$, $\tan t$, $\frac{1}{1+t^2}$, $\frac{\sin t}{t}$, etc.

6. a. $3^2+4^2=25$

$R=5 \qquad \delta=\tan^{-1}\left(\frac{-4}{3}\right)$

$5\cos[2(t+\tan^{-1}(4/3))] \approx 5\cos(2t+1.85459)$

b. $R=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}$

$\delta=\tan^{-1}\left(\frac{3}{2}\right) \approx .982794$

$\sqrt{13}\cos(\pi(t-\tan^{-1}(\tfrac{3}{2}))) \approx \sqrt{13}\cos(\pi t-3.08754)$

7. $.2q''+Rq'+\frac{1}{.8\times10^{-6}}q=0$

$.2r^2+Rr+1.25\times10^6=0$

$r^2+.5Rr+6.25\times10^6$

$r=\dfrac{-5R\pm\sqrt{25R^2-4(6.25\times10^6)}}{2}$

Critically damped $\Rightarrow$

$25R^2-4(6.25\times10^6)=0$

$25R^2=2.5\times10^7$

$R^2=10^6$

$\boxed{R=1000}$

8. $.2q''+300q'+10^5q=0 \qquad q(0)=10^{-6},\ q'(0)=0$

$q''+1500q'+5\times10^5q=0$

$\boxed{q(t)=2\times10^{-6}e^{-500t}-10^{-6}e^{-1000t}}$

$r^2+1500r+5\times10^5=0$

$\dfrac{-1500\pm\sqrt{1500^2-5\times10^5\times4}}{2}=\dfrac{-1500\pm500}{2}=-500,\ -1000$

$q(t)=c_1e^{-500t}+c_2e^{-1000t} \qquad q(0)=c_1+c_2=10^{-6}$

$q'(t)=-500c_1e^{-500t}-1000e^{-1000t} \qquad q'(0)=-500c_1-1000c_2=0$

$1000c_1+1000c_2=10^{-3}$

$-\ 500c_1-1000c_2=0$

$500c_1=10^{-3}$

$c_1=2\times10^{-6} \qquad c_2=-1\times10^{-6}$