Math 2255 Homework #7 Key

1. $f(t) = \cancel{0 - 0\,\mathcal{U}(t-1)} + [(t-1)^2 - 2(t-1) + 2]\,\mathcal{U}(t-1) =$

$$\mathcal{L}^{-1}\{f(t)\} = e^{-s}\left(\frac{2}{s^3} - \frac{2}{s^2} + \frac{2}{s}\right) = 2e^{-s}\left(\frac{1-s+s^2}{s^3}\right)$$

2\. a. $\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} = e^{2t}t^3$

b. $\frac{1}{2}\mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{2}{s^2-4}\right\} = \frac{1}{2}\sinh 2(t-2)\,\mathcal{U}(t-2)$

c. $\mathcal{L}^{-1}\left\{\frac{(s-2)e^{-s}}{s^2-4s+3}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{s-2}{(s-2)^2-1}\right)\right\} = e^{2t}\cosh(t-1)\,\mathcal{U}(t-1)$

3a. $\mathcal{L}\left\{\int_0^t (t-\tau)^2 \cos 2\tau \, d\tau\right\}$

$f(\tau) = \cos 2\tau$
$g(t-\tau) = (t-\tau)^2 \Rightarrow g(t) = t^2$

$$= \mathcal{L}\{\cos 2t\} \cdot \mathcal{L}\{t^2\} = \frac{\cancel{s}}{s^2+4} \cdot \frac{2}{s^{\cancel{3}2}} = \frac{2}{s^2(s^2+4)}$$

b. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} \cdot \frac{s}{s^2+4}\right\} =$

$f(t) = e^{-t}$
$g(\tau) = \cos 2t$

$$\int_0^t e^{t-\tau}\cos 2\tau \, d\tau$$

4a. $y'' + 2y' + 2y = \sin \alpha t$, $y(0) = 0$, $y'(0) = 0$

$$s^2Y(s) + 2sY(s) + 2Y(s) = \frac{\alpha}{s^2+\alpha^2}$$

$$Y(s)(s^2+2s+2) = \frac{\alpha}{s^2+\alpha^2}$$

$$Y(s) = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{\underbrace{s^2+2s+2}_{(s^2+2s+1)+1}} = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}$$

$f(t) = \sin \alpha t$
$g(t) = e^{-t}\sin t$

$$y(t) = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin \alpha\tau \, d\tau \text{ or } \int_0^t e^{-\tau}\sin\tau \sin\alpha(t-\tau)\, d\tau$$

②

4b. $y^{IV} - y = g(t)$, $y(0)=0$, $y'(0)=0$, $y''(0)=0$, $y'''(0)=0$

$$s^4 Y(s) - Y(s) = G(s)$$

$$Y(s)(s^4-1) = G(s) \Rightarrow Y(s) = G(s)\cdot\frac{1}{s^4-1} =$$

$$G(s)\left[\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-1}\right]$$

$$(As+B)(s^2-1) + (Cs+D)(s^2+1) = 1$$

$s=1$ $\quad (C+D)2=1 \Rightarrow C+D=\frac{1}{2}$
$s=-1$ $\quad (-C+D)2=1 \quad -C+D=\frac{1}{2}$ $\quad > 2D=1 \Rightarrow D=\frac{1}{2} \Rightarrow C=0$

$s=0$ $\quad -B+D=1 \Rightarrow -B=\frac{1}{2} \Rightarrow B=\frac{1}{2}$

$s=2$ $\quad (2A+B)(3)+(2C+D)(5)=1 \Rightarrow 6A+\frac{3}{2}+\frac{5}{2}=1 \Rightarrow 6A=-3$
$\Rightarrow A=-\frac{1}{2}$

$$Y(s) = G(s)\cdot\left[-\frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2-1}\right]$$

$$\mathcal{L}^{-1}\left\{-\frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2-1}\right] = -\frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}\sinh t$$

$$y(t) = \frac{1}{2}\int_0^t g(t-\tau)(\sin\tau - \cos\tau + \sinh\tau)\,d\tau$$

5. $\varphi'(t) + \varphi(t) = \int_0^t \sin(t-\xi)\,\varphi(\xi)\,d\xi \quad \varphi(0)=1$

$$s\Phi(s) - 1 + \Phi(s) = \frac{1}{s^2+1}\cdot\Phi(s)$$

$$\Phi(s)\left(s+1-\frac{1}{s^2+1}\right) = 1 = \left(\frac{(s+1)(s^2+1)-1}{s^2+1}\right)\Phi(s) = \left(\frac{s^3+s^2+s}{s^2+1}\right)\Phi(s)$$

$$\Phi(s) = \frac{s^2+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$A(s^2+s+1) + (Bs+C)(s) = s^2+1 \qquad s=0 \quad A=1$$

$s=1$ $\quad 3A+(B+C)=2 \Rightarrow B+C=-1$

$s=-1$ $\quad A+(-B+C)(-1)=2 \qquad B-C=1$

$B=0 \Rightarrow C=-1$

5 contd. ③

$\Phi(s) = \frac{1}{s} - \frac{1}{s^2+s+1} = \frac{1}{s} - \frac{1}{(s^2+s+\frac{1}{4})+\frac{3}{4}} = \frac{1}{s} - \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}$

$k = \frac{\sqrt{3}}{2}$

$\varphi(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}/2}{(s+1/2)^2+\frac{3}{4}}\right\} =$

$1 - \frac{2}{\sqrt{3}} e^{-1/2 t} \sin\left(\frac{\sqrt{3}}{2}t\right)$

6. a. $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$ $\quad \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x^n)} \right| = \lim_{n\to\infty} \underbrace{\left|\frac{n+1}{n}\right|}_{=1} \cdot \left|\frac{x}{2}\right| < 1$

$-1 < \frac{x}{2} < 1 \Rightarrow -2 < x < 2$

converges on $(-2,2)$
diverges at both endpoints

b. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ $\quad \lim_{n\to\infty} \left| \frac{x^{2n+2}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| < 1$ for all x

$(-\infty, \infty)$

c. $\sum_{n=1}^{\infty} \frac{n!\, x^n}{n^n}$ $\quad \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}} \cdot \frac{n^n}{n!\, x^n} \right| = \lim_{n\to\infty} \underbrace{\left|\frac{n^n}{(n+1)^n}\right|}_{=1} \cdot \underbrace{\left|\frac{(n+1)!}{(n+1)n!}\right|}_{=1} \cdot |x| < 1$

$-1 < x < 1$

7. a. $\sin x,\ x_0 = 0$

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

b. $\ln(x),\ x_0 = 1$

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

c. $\frac{1}{1+x},\ x_0 = 0$ $\quad \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

7d. $\cosh(x)$, $x_0 = 0$ $\quad \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

you can derive these, or look them up in a calc book.

8. $f(x) = \sum_{n=0}^{\infty} a_n x^n$

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n\, x^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)\, x^n$$

$$f''(x) = \sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\, x^n$$

$$f'''(x) = \sum_{n=3}^{\infty} a_n(n)(n-1)(n-2)\, x^{n-3} = \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)(n+1)x^n$$

$$f^{iv}(x) = \sum_{n=4}^{\infty} a_n(n)(n-1)(n-2)(n-3)\, x^{n-4} = \sum_{n=0}^{\infty} a_{n+4}(n+4)(n+3)(n+2)(n+1)\, x^n$$

9. a. $x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n + a_0 = \sum_{n=1}^{\infty} a_n(n+1)x^n + a_0$$

b. $x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=1}^{\infty} (n+1)(n)a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (n+1)(n)a_{n+1} x^n + \sum_{n=1}^{\infty} a_n x^n + a_0$$

$$= \sum_{n=1}^{\infty} \left[(n+1)(n)\,a_{n+1} + a_n\right] x^n + a_0$$