1. $(1-x)y'' + y = 0$, $x_0 = 0$

$$y'' - xy'' + y = 0 \Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n+1}(n+1)(n)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$a_2(2)(1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} a_{n+1}(n+1)(n)x^n + a_0x^0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$2a_2 + a_0 = 0 \qquad \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) - a_{n+1}(n+1)(n) + a_n\right]x^n = 0$$

$a_2 = -\frac{1}{2}a_0$

$n=0$

$$a_{n+2} = \frac{a_{n+1}\cancel{(n+1)}n}{(n+2)\cancel{(n+1)}} - \frac{a_n}{(n+2)(n+1)}$$

$$a_{n+2} = \frac{a_{n+1}(n)}{n+2} - \frac{a_n}{(n+2)(n+1)}$$

$n=1$ $\quad a_3 = \frac{a_2(1)}{3} - \frac{a_1}{(3)(2)} = \frac{1}{3}\left(-\frac{1}{2}a_0\right) - \frac{1}{6}a_1 = -\frac{1}{6}a_0 - \frac{1}{6}a_1$

$n=2$

$a_4 = \frac{a_3\cancel{(2)}}{\cancel{4}\,2} - \frac{a_2}{4\cdot 3} = \frac{1}{2}\left(-\frac{1}{6}a_0 - \frac{1}{6}a_1\right) - \frac{1}{12}\left(-\frac{1}{2}a_0\right) =$

$-\frac{1}{12}a_0 - \frac{1}{12}a_1 + \frac{1}{24}a_0 =$

$-\frac{1}{24}a_0 - \frac{1}{12}a_1$

$n=3$ $\quad a_5 = \frac{a_4(3)}{5} - \frac{a_3}{5\cdot 4} = \frac{3}{5}\left(-\frac{1}{24}a_0 - \frac{1}{12}a_1\right) - \frac{1}{20}\left(-\frac{1}{6}a_0 - \frac{1}{6}a_1\right)$

$-\frac{1}{40}a_0 - \frac{1}{20}a_1 + \frac{1}{120}a_0 + \frac{1}{120}a_1 =$

$-\frac{1}{60}a_0 - \frac{1}{24}a_1$

$$y = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 + \dots\right) +$$
$$a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \dots\right)$$

16. $xy'' + y' + xy = 0 \quad x_0 = 1$ ②

$(x-1)y'' + y'' + y' + (x-1)y + y = 0$

$(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} +$

$(x-1)\sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1}$

$+ \sum_{n=0}^{\infty} a_n (x-1)^n = 0$

$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+3}(n+3)(n+2)(x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+2}(n+2)(x-1)^{n+1}$

$+ \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=-1}^{\infty} a_{n+1}(x-1)^{n+1} = 0$

$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^{n+1} + a_2(2)(1)x^0 + \sum_{n=0}^{\infty} a_{n+3}(n+3)(n+2)(x-1)^{n+1} +$

$a_1(1)x^0 + \sum_{n=0}^{\infty} a_{n+2}(n+2)(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + a_0 x^0 + \sum_{n=0}^{\infty} a_{n+1}(x-1)^{n+1} = 0$

$2a_2 + a_1 + a_0 = 0 \implies a_2 = -\frac{1}{2}a_1 - \frac{1}{2}a_0$

$\sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) + a_{n+3}(n+3)(n+2) + a_{n+2}(n+2) + a_n + a_{n+1} \right] x^{n+1} = 0$

$a_{n+3}(n+3)(n+2) + a_{n+2}\underbrace{\left[(n+2)(n+1) + (n+2)\right]}_{(n+2)(n+2)} + a_n + a_{n+1} = 0$

$a_{n+3} = \frac{-a_{n+2}(n+2)^{\cancel{2}}}{(n+3)\cancel{(n+2)}} - \frac{a_{n+1}}{(n+3)(n+2)} - \frac{a_n}{(n+3)(n+2)}$

$n=0$

$a_3 = \frac{-a_2(2)}{3} - \frac{a_1}{3\cdot 2} - \frac{a_0}{3\cdot 2} = -\frac{2}{3}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) - \frac{1}{6}a_1 - \frac{1}{6}a_0 =$

$\frac{1}{3}a_1 - \frac{1}{6}a_1 + \frac{1}{3}a_0 - \frac{1}{6}a_0 = \frac{1}{6}a_1 + \frac{1}{6}a_0$

$n=1$

$a_4 = \frac{-a_3(3)}{4} - \frac{a_2}{4\cdot 3} - \frac{a_1}{4\cdot 3} = -\frac{3}{4}\left(\frac{1}{6}a_1 + \frac{1}{6}a_0\right) - \frac{1}{12}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right) - \frac{1}{12}a_1 =$

1b. cont'd.

$-\frac{1}{8}a_1 - \frac{1}{8}a_0 + \frac{1}{24}a_1 + \frac{1}{24}a_0 - \frac{1}{12}a_1 = -\frac{1}{6}a_1 - \frac{1}{12}a_0$

$n=2$ $a_5 = -\frac{a_4(4)}{5} - \frac{a_3}{5\cdot 4} - \frac{a_2}{5\cdot 4} = -\frac{4}{5}\left(-\frac{1}{6}a_1 - \frac{1}{12}a_0\right) - \frac{1}{20}\left(\frac{1}{6}a_1 + \frac{1}{6}a_0\right) - \frac{1}{20}\left(-\frac{1}{2}a_1 - \frac{1}{2}a_0\right)$

$= \frac{2}{15}a_1 + \frac{1}{15}a_0 - \frac{1}{120}a_1 - \frac{1}{120}a_0 + \frac{1}{40}a_1 + \frac{1}{40}a_0 = \frac{3}{20}a_1 + \frac{1}{12}a_0$

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{12}x^5 + \cdots\right)$$
$$+ a_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{3}{20}x^5 + \cdots\right)$$

1c. $(2+x^2)y'' + xy' + 4y = 0 \qquad y(0) = -1, \ y'(0) = 3$

$$2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^2\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n\, x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\sum_{n=0}^{\infty} 2a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$2(a_2)(2)(1)x^0 + 2a_3(3)(2)x^1 + \sum_{n=2}^{\infty} 2a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^n +$$
$$a_1(1)x^1 + \sum_{n=2}^{\infty} a_n n x^n + 4a_0x^0 + 4a_1x^1 + \sum_{n=2}^{\infty} 4a_n x^n = 0$$

$4a_2 + 4a_0 = 0 \Rightarrow a_2 = -a_0 \qquad 12a_3 + a_1 + 4a_1 = 0 \Rightarrow a_3 = \frac{-5}{12}a_1$

$$\sum_{n=2}^{\infty}\left[2a_{n+2}(n+2)(n+1) + a_n n(n-1) + a_n n + 4a_n\right]x^n = 0$$

$a_n n(n-1+1) = a_n(n^2+4)$

$$2a_{n+2}(n+2)(n+1) = -a_n(n^2+4)$$

$$a_{n+2} = \frac{-a_n(n^2+4)}{2(n+2)(n+1)}$$

$n=2$

$$a_4 = \frac{-a_2(8)}{\cancel{2(4)}(3)} = -\frac{1}{3}(-a_0) = \frac{1}{3}a_0$$

1c. cont'd

$n=3$ $a_5 = \dfrac{-a_3(13)}{2(5)(3)} = -\dfrac{13}{\cancel{30}_6}\left(-\dfrac{\cancel{5}}{12}a_1\right) = \dfrac{13}{72}a_1$

$n=4$ $a_6 = \dfrac{-a_4(\cancel{20}^{42})}{\cancel{2}(\cancel{6}_3)(8)} \quad -\dfrac{1}{3}\left(\dfrac{1}{3}a_0\right) = -\dfrac{1}{9}a_0$

$n=5$ $a_7 = \dfrac{-a_5(29)}{2(7)(6)} = -\dfrac{29}{84}\left(\dfrac{13}{72}a_1\right) = \dfrac{-377}{6048}a_1$

$n=6$ $a_8 = \dfrac{-a_6(\cancel{40}^{5})}{2(\cancel{8})(7)} = -\dfrac{5}{14}\left(-\dfrac{1}{9}a_0\right) = \dfrac{5}{126}a_0$

$n=7$ $a_9 = \dfrac{-a_7(53)}{2(9)(8)} = \dfrac{-53}{144}\left(\dfrac{-377}{6048}a_1\right) = \dfrac{19,981}{870,912}a_1$

$$y = a_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{9}x^6 + \frac{5}{126}x^8 + \dots\right) +$$
$$a_1\left(x - \frac{5}{12}x^3 + \frac{13}{72}x^5 - \frac{377}{6048}x^7 + \frac{19,981}{870,912}x^9 + \dots\right)$$

$a_0 = -1, \quad a_1 = 3$

$$\Rightarrow y = -1 + 3x + x^2 - \frac{5}{4}x^3 - \frac{1}{3}x^4 + \frac{13}{24}x^5 + \frac{1}{9}x^6 - \frac{377}{2016}x^7 + \dots$$

2. $y'' + x^2y = 0, \quad y(0) = 1, \quad y'(0) = 0 \quad \to a_1 = 0$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=-2}^{\infty} a_{n+4}(n+4)(n+3)x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$a_2(2)(1)x^0 + a_3(3)(2)x^1 + \sum_{n=0}^{\infty} a_{n+4}(n+4)(n+3)x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$2a_2 = 0 \quad 6a_3 = 0 \quad \displaystyle\sum_{n=0}^{\infty}\left[a_{n+4}(n+4)(n+3) + a_n\right]x^{n+2} = 0$

2 cont'd. ⑤

$a_2 = 0, \ a_3 = 0$

$$a_{n+4} = \frac{-a_n}{(n+4)(n+3)}$$

$n=0$ $a_4 = \frac{-a_0}{4 \cdot 3} = -\frac{1}{12}a_0$ $n=1$ $a_5 = \frac{-a_1}{5 \cdot 4} = -\frac{1}{20}a_1$

$n=2$ $a_6 = \frac{-a_2}{6 \cdot 5} = -\frac{1}{30}(0) = 0$ $n=3$ $a_7 = \frac{-a_3}{7 \cdot 6} = -\frac{1}{42}(0)$

$n=4$ $a_8 = \frac{-a_4}{8 \cdot 7} = \frac{-1}{56}\left(-\frac{1}{12}a_0\right) = \frac{1}{672}a_0$ $n=5$ $a_9 = \frac{-a_5}{9 \cdot 8} = \frac{-1}{72}\left(-\frac{1}{20}a_1\right) = \frac{1}{144}a_1$

$n=6$ $a_{10} = 0$ $n=7$ $a_{11} = 0$

$n=8$ $a_{12} = \frac{-a_8}{12 \cdot 11} = \frac{-1}{132}\left(\frac{1}{672}\right)a_0 = \frac{-1}{88,704}a_0$ $n=9$ $a_{13} = \frac{-a_9}{13 \cdot 12} = \frac{-1}{156}\left(\frac{1}{44}\right) = \frac{-1}{22,464}a_1$

$$y = a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \dots\right) + a_1\left(x - \frac{1}{20}x^5 + \frac{1}{144}x^9 - \frac{1}{22,464}x^{13} + \dots\right)$$

$a_0 = 1$

$a_1 = 0$

$a_{16} = \frac{-a_{12}}{16 \cdot 15} = \frac{+1}{21,288,960}a_0$ $a_{20} = \frac{-a_{16}}{20 \cdot 19} = \frac{-1}{8,089,804,800}a_0$

graphing $Y_1 = 1$

$Y_2 = 1 - \frac{1}{12}x^4$

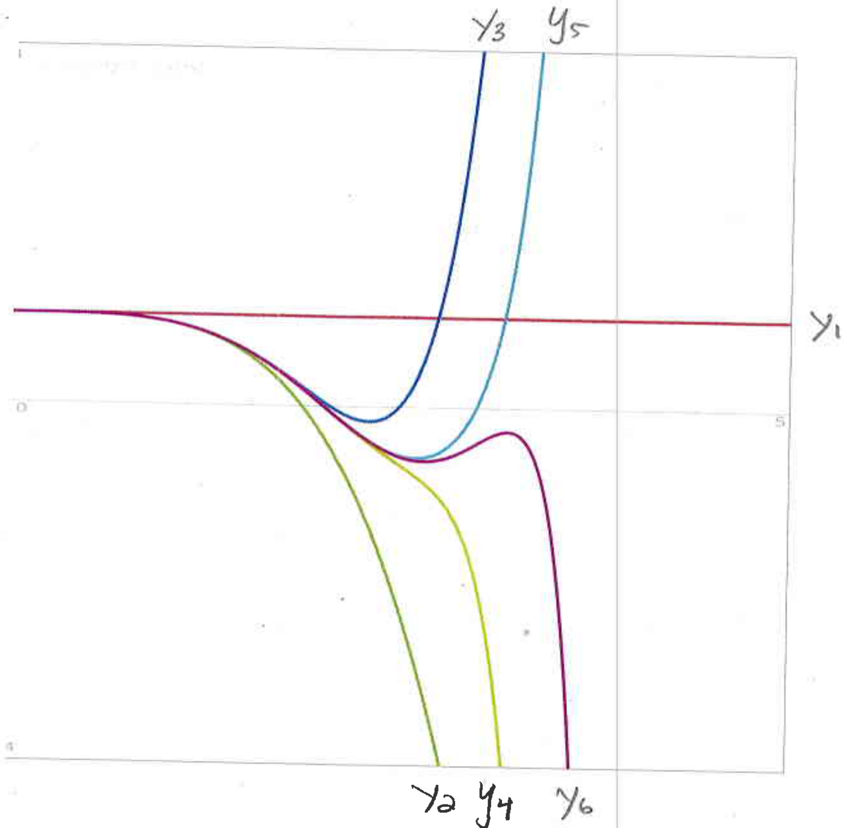
$Y_3 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8$

$Y_4 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12}$

$Y_5 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \frac{1}{21,288,960}x^{16}$

$Y_6 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \frac{1}{88,704}x^{12} + \frac{1}{21,288,960}x^{16} - \frac{1}{8,089,804,800}x^{20}$

See attached graph



y_3
y_5
y_1
0
5
y_2
y_4
y_6

3. an ordinary point of the standard differential equation $y^{(n)} + P_1(x) y^{(n-1)} + \cdots P_{(n-2)}(x) y' + q(x) y = 0$ is defined for all $p_i(x)$ and $q(x)$ in the equation.

a singular point is a point which is not defined for some $p_i(x)$ or $q(x)$.

in #1a, $x=1$ is a singular point, and in 2b, $x=0$ is a singular point. All x values in #1c are ordinary.

4 a. $y' - y = 0$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+1}(n+1) x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[a_{n+1}(n+1) - a_n\right] x^n = 0 \Rightarrow \quad a_{n+1} = \frac{a_n}{n+1}$$

$n=0$ $\quad a_1 = \frac{a_0}{1} \qquad n=1 \quad a_2 = \frac{a_1}{2} = \frac{1}{2} a_0 \qquad n=2 \quad a_3 = \frac{a_2}{3} = \frac{1}{3}\left(\frac{1}{2}a_0\right) = \frac{1}{6} a_0$

$n=3$ $\quad a_4 = \frac{a_3}{4} = \frac{1}{4}\left(\frac{1}{6}a_0\right) = \frac{1}{24} a_0 \qquad a_{n+1} = \frac{a_0}{n!}$

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = a_0 e^x$$

b. $y' + xy = 1 + x$

$$\sum_{n=1}^{\infty} a_n(n) x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = 1 + x$$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 1 + x$$

$$\sum_{n=-1}^{\infty} a_{n+2}(n+2) x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 1 + x$$

$$a_1(1)x^0 + \sum_{n=0}^{\infty} a_{n+2}(n+2) x^{n+1} + \sum_{n=0} a_n x^{n+1} = 1 + x$$

4b cont'd

$a_1 = 1 \qquad \sum_{n=0}^{\infty} \left[a_{n+2}(n+2) + a_n\right] x^{n+1} = x$

$$\sum_{n=1}^{\infty} [a_{n+2}(n+2) + a_n] x^{n+2} = 0 \qquad \left(a_2(2) + a_0\right)x^1 = x$$

$$a_{n+2} = \frac{-a_n}{n+2} \qquad 2a_2 + a_0 = 1$$

$$a_2 = \frac{1-a_0}{2} = \frac{1}{2} - \frac{1}{2}a_0$$

$n=1$ $\quad a_3 = \frac{-a_1}{3} = -\frac{1}{3}$ $\quad n=2$ $\quad a_4 = \frac{-a_2}{4} = -\frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}a_0\right) = -\frac{1}{8} + \frac{1}{8}a_0$

$n=3$ $\quad a_5 = \frac{-a_3}{5} = -\frac{1}{5}\left(-\frac{1}{3}\right) = \frac{1}{15}$ $\quad n=4$ $\quad a_6 \ \frac{-a_4}{6} = -\frac{1}{6}\left(-\frac{1}{8} + \frac{1}{8}a_0\right) = \frac{1}{48} - \frac{1}{48}a_0$

$n=5$ $\quad a_7 = \frac{-a_5}{7} = -\frac{1}{7}\left(\frac{1}{15}\right) = \frac{1}{105}$ $\quad n=6$ $\quad a_8 = \frac{-a_6}{8} = -\frac{1}{8}\left(\frac{1}{48} - \frac{1}{48}a_0\right) = -\frac{1}{384} + \frac{1}{384}a_0$

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 \ldots\right) +$$

$$\left(x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{15}x^5 + \frac{1}{48}x^6 + \frac{1}{105}x^7 - \frac{1}{384}x^8 + \cdots\right)$$

$$\mu = e^{\int x\,dx} = e^{x^2/2}$$

$$e^{x^2/2}y' + xe^{x^2/2}y = (1+x)e^{x^2/2}$$

$$\int\left(e^{x^2/2}y\right)' = \int(1+x)e^{x^2/2}$$

$$y = e^{-x^2/2}\int(1+x)e^{x^2/2}\,dx = e^{-x^2/2}\int e^{x^2/2}\,dx + e^{-x^2/2}\left(e^{x^2/2} + C\right)$$

$$= e^{-x^2/2}\int e^{x^2/2}\,dx + 1 + Ce^{-x^2/2}$$

Computing this integral has to be done by Taylor series

5\.

a regular singular point in a second order equation (8)

$y'' + p(x)y' + q(x)y = 0$ has a zero in $p(x)$ at that point x_0, but $(x-x_0)p(x)$ does not. and $q(x)$ may have such a point at x_0, but $(x-x_0)^2 q(x)$ does not.

This is generalized to a higher order equation, by

$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots a_1(x)y' + a_0(x)y = 0$

So that $a_i(x)$ may have a zero at the point x_0 but

$a_i(x)(x-x_0)^{n-i}$ does not.

for example $\quad y'' + \frac{1}{x(x+1)}y' - \frac{1}{x^2(x+1)^3}y = 0$

has a regular singular point at $x=0$, but an irregular one at $x=-1$.

6\. a. $\dfrac{x^2(1-x)^2y'' + 2xy' + 4y = 0}{x^2(1-x)^2}$

$$y'' + \frac{2\cancel{x}}{x^{\cancel{2}}(1-x)^2}y' + \frac{4}{x^2(1-x)^2}y = 0$$

$$y'' + \frac{2}{x(1-x)^2}y' + \frac{4}{x^2(1-x)^2}y = 0$$

$x=0$ is a regular singular point, but $x=1$ is irregular.

b. $\dfrac{x(1-x^2)^3y'' + (1-x^2)^2y' + 2(1+x)y = 0}{x(1-x^2)^3}$

$$y'' + \frac{1}{x(1-x^2)}y' + \frac{2}{x(1-x)^3(1+x)^2}y = 0$$

regular at $x=0$ and $x=-1$
but irregular at $x=1$

6c. $\dfrac{(x^2+x-2)y'' + (x+1)y' - 2y = 0}{(x+2)(x-1)}$

$$y'' + \frac{(x+1)}{(x+2)(x-1)}y' - \frac{2}{(x+2)(x-1)}y = 0$$

$x=-2$ and $x=1$ are both regular singular points

d. $\dfrac{xy'' + y' + (\cot x)y = 0}{x}$

$$y'' + \frac{1}{x}y' + \left(\frac{\cot x}{x}\right)y = 0$$

Singular points $x=0, \pi, \pm k\pi$ are all irregular

7a. $2x^2\sum_{n=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2} - 4x\sum_{n=1}^{\infty} a_n(n+r)x^{n+r-1} + 6\sum_{n=0}^{\infty} a_n x^{n+r} = 0$

$$\sum_{n=2}^{\infty} 2a_n(n+r)(n+r-1)x^n - \sum_{n=1}^{\infty} 4a_n(n+r)x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2a_n(n+r)(n+r-1)x^n - 4a_1 r x^1 - \sum_{n=2}^{\infty} 4a_n(n+r)x^n + 6a_0x^0 + 6a_1x^1 + \sum_{n=2}^{\infty} 6a_n x^n = 0$$

$-4a_1 r + 6a_1 = 0$

$a_1(-4r+6) = 0$

$r = \frac{6}{4} = \frac{3}{2}$

$$\sum_{n=2}^{\infty}\left[2a_n(n+r)(n+r-1) - 4a_n(n+r) + 6a_n\right]x^n = 0$$

$$2a_n\left[(n+3/2)(n+1/2) - 2(n+3/2) + 3\right] = 0$$

$n^2 + \frac{1}{2}n + 3/2 n + \frac{3}{4} - 2n - \cancel{3} + \cancel{3} =$

$n^2 \cancel{+2n} \cancel{-2n} + 3/4 = n^2 + 3/4 = 0$

$n^2 = -3/4$

$n = \pm\frac{\sqrt{3}i}{2}$

$n + r = \frac{3}{2} \pm \frac{\sqrt{3}i}{2}$

Cauchy-Euler

$2(n)(n-1) - 4(n) + 6 = 0 \Rightarrow 2n^2 - \cancel{2n} - \cancel{4n} + 6 = 2n^2 + 6 = n^2 = -3$

$n = \pm\sqrt{3}i$ $\qquad y = C_1\cos(\ln\sqrt{3}t) + C_2\sin(\ln\sqrt{3}t)$

contd.

7a. The series solution is harder to interpret w/ complex powers.

7b. $(x-2)^2 y'' + 5(x-2)y' + 8y = 0 \quad x_0 = 2$

$$(x-2)^2 \sum_{n=2}^{\infty} a_n (n+r)(n+r-1)(x-2)^{n+r-2} + 5(x-2)\sum_{n=1}^{\infty} a_n (n+r)(x-2)^{n+r-1} +$$

$$8 \sum_{n=0}^{\infty} a_n (x-2)^{n+r} = 0$$

$$\sum_{n=2}^{\infty} a_n (n+r)(n+r-1)(x-2)^n + \sum_{n=1}^{\infty} 5a_n (n+r)(x-2)^n + \sum_{n=0}^{\infty} 8a_n (x-2)^n = 0$$

$$\sum_{n=2}^{\infty} a_n (n+r)(n+r-1)(x-2)^n + 5a_1(1+r)(x-2)^1 + \sum_{n=2}^{\infty} 5a_n (n+r)(x-2)^n +$$

$$8a_0 (x-2)^0 + 8a_1 (x-2)^1 + \sum_{n=2}^{\infty} 8a_n (x-2)^n = 0$$

$$5a_1(1+r) + 8a_1 = 0$$

$$a_1 (5 + 5r + 8) = 0$$

$$5r + 13 = 0$$

$$r = -\frac{13}{5}$$

$$\sum_{n=2}^{\infty} \left[a_n (n+r)(n+r-1) + 5a_n (n+r) + 8a_n\right](x-2)^n = 0$$

$$a_n\left[\left(n-\tfrac{13}{5}\right)\left(n-\tfrac{18}{5}\right) + 5\left(n-\tfrac{13}{5}\right) + 8\right] = 0$$

$$\left[n^2 - \tfrac{18}{5}n - \tfrac{13}{5}n + \tfrac{234}{25} + 5n - 13 + 8\right]$$

$$25n^2 - 155n + 234 + 125n - 325 + 200$$

$$25n^2 - 30n + 109 = 0$$

$$n = \frac{30 \pm \sqrt{900 - 4(25)(109)}}{100} = \frac{30 \pm 100i}{100} = \frac{3}{10} \pm i$$

$$(x-2)^{n+r} = (x-2)^{-13/5 + \frac{3}{10} \pm i} = (x-2)^{-23/10 \pm i}$$

hard to interpret complex powers.

Cauchy-Euler.

$$n(n-1) + 5n + 8 = 0 \Rightarrow n^2 - n + 5n + 8 = n^2 + 4n + 8 = 0$$

$$n = \frac{-4 \pm \sqrt{16 - 4(8)}}{2} = \frac{-4 \pm 4i}{2} = -2 \pm 2i$$

$$y = c_1 x^{-2} \cos(\ln 2x) + c_2 x^{-2} \sin(\ln 2x)$$