Differential Equations Hyperbolic Trig Functions Key

1. $\frac{d}{dx}[\tanh(x)] = \operatorname{sech}^2 x$

$$\frac{d}{dx}[\tanh(x)] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\frac{d}{dx}(\sinh x)\cosh x - \frac{d}{dx}(\cosh x)\sinh x}{\cosh^2 x} =$$

$$\frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}[\tanh(x)] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right] = \frac{\frac{d}{dx}[e^x - e^{-x}](e^x + e^{-x}) - \frac{d}{dx}[e^x + e^{-x}](e^x - e^{-x})}{(e^x + e^{-x})^2} =$$

$$\frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = \operatorname{sech}^2 x$$

2. $$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} =$$

$$\frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4} = \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4} = \frac{4}{4}$$

$$= 1$$

3. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \qquad \cosh(x) = e^x + e^{-x}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

$$2 + 2\left(\frac{x^2}{2!}\right) + 2\left(\frac{x^4}{4!}\right) + \ldots = e^x + e^{-x}$$

$$\frac{e^x+e^{-x}}{2} = \frac{2+2\left(\frac{x^2}{2!}\right)+2\left(\frac{x^4}{4!}\right)+\cdots}{2} \qquad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$$

②

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

4. $\int \sinh(x)\,dx = \cosh(x)+C$ since $\frac{d}{dx}[\cosh x] = \sinh x$

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$$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|\cosh x|+C$$

$u = \cosh x$
$du = \sinh x\,dx$

$\boxed{\text{no negative here}}$

$$\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|\sinh x|+C$$

$u = \sinh x$
$du = \cosh x\,dx$

$\int \operatorname{sech} x\,dx$

As it turns out, there are no parallel counterparts for $\int \operatorname{sech} x\,dx$ and $\int \operatorname{csch} x\,dx$

instead, they have to be obtained through various identities and in the case of $\int \operatorname{sech} x$, trig substitutions

$$\int \operatorname{sech} x\,dx = \sin^{-1}(\tanh x)+C$$

$$\frac{d}{dx}[\sin^{-1}(\tanh x)] = \frac{1}{\sqrt{1-\tanh^2 x}}\cdot \operatorname{sech}^2 x = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$$

$$\int \operatorname{csch} x\,dx = \ln\left|\tanh \frac{x}{2}\right|+C$$

$$\frac{d}{dx}\left[\ln\left|\tanh \frac{x}{2}\right|\right] = \frac{1}{\tanh \frac{x}{2}}\cdot \frac{1}{2}\operatorname{sech}^2 \frac{x}{2} = \frac{1}{2}\left[\frac{\cosh x/2}{\sinh x/2}\right]\left[\frac{1}{\cosh \frac{x}{2}}\right]^2 =$$

$$\frac{1}{2}\left[\frac{1}{\sinh \frac{x}{2}\cosh \frac{x}{2}}\right] = \frac{1}{2\sinh \frac{x}{2}\cosh \frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{csch} x$$

to derive these formulas, you have to work the steps backwards, which is much more opaque.

5.a. $y'' - 4y = 0$

$r^2 - 4 = 0$ $\qquad y_1 = Ae^{-2t} \quad y_2 = Be^{2t}$

$r = 2,\ r = -2$

OR

$y_1 = C\sinh 2t, \quad y_2 = D\cosh 2t$

$y(t) = Ae^{-2t} + Be^{2t}$

$y' = -2Ae^{-2t} + 2Be^{2t}$

$y'' = 4Ae^{-2t} + 4Be^{2t}$

$4Ae^{-2t} + 4Be^{2t} - 4(Ae^{-2t} + Be^{2t}) = 4Ae^{-2t} + 4Be^{2t} - Ae^{-2t} - 4Be^{2t} = 0$ ✓

$y(t) = C\sinh 2t + D\cosh 2t$

$y'(t) = 2C\cosh 2t + 2D\sinh 2t$

$y''(t) = 4C\sinh 2t + 4D\cosh 2t$

$4C\sinh 2t + 4D\cosh 2t - 4(C\sinh 2t + D\cosh 2t) =$

$4C\sinh 2t + 4D\cosh 2t - 4C\sinh 2t - 4D\cosh 2t = 0$ ✓

b. $y'' - 9y = 0$ $\qquad r^2 - 9 = 0 \qquad r = 3,\ r = -3$

$y_1(t) = Ae^{-3t},\ y_2(t) = Be^{3t}$ OR $y_1(t) = C\sinh 3t,\ y_2(t) = D\cosh 3t$

$y(t) = Ae^{-3t} + Be^{3t}$

$y'(t) = -3Ae^{-3t} + 3Be^{3t} \qquad y''(t) = 9Ae^{-3t} + 9Be^{3t}$

$9Ae^{-3t} + 9Be^{-3t} - 9(Ae^{-3t} + Be^{3t}) = 0$ ✓

$y(t) = C\sinh 3t + D\cosh 3t \qquad y'(t) = 3C\cosh 3t + 3D\sinh 3t$

$y''(t) = 9C\sinh 3t + 9D\cosh 3t$

$9C\sinh 3t + 9D\cosh 3t - 9(C\sinh 3t + D\cosh 3t) = 0$ ✓

6. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} \frac{(2n+1)\,x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \cosh x$$

$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right] = \sum_{n=1}^{\infty} \frac{(2n)\,x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \text{ (re-index)} =$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$