Name KEY

Math 2568, Exam #2 – Part 1, Spring 2013

Instructions: On this portion of the exam, you may **NOT** use a calculator. Show all work. Answers must be supported by work to receive full credit.

1. Compute the determinant by the cofactor method. (10 points)

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 1 \\ 3 & 1 & 5 & -1 \end{vmatrix}$$

$$-5\begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{vmatrix} = -5\left[2\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\right] =$$

$$-5\left[2(2-2) + 1(1-1)\right] =$$

$$-5\left[2(0) + 1(0)\right] = -5(0) = 0$$

2. Compute the determinant by using row operations. (7 points)

$$\begin{vmatrix} 0 & 3 & -1 & 5 \\ 1 & 0 & -2 & 4 \\ -3 & 2 & 1 & -3 \\ 0 & 5 & 2 & 3 \end{vmatrix}$$

exchange $R_1 \leftrightarrow R_2$ (-1) change $\rightarrow \begin{vmatrix} 1 & 0 & -2 & 4 \\ 0 & 3 & -1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & 5 & 2 & 3 \end{vmatrix}$ $3R_1 + R_3 \rightarrow R_3$ no change $\begin{vmatrix} 1 & 0 & -2 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 2 & -5 & 9 \\ 0 & 5 & 2 & 3 \end{vmatrix}$

$\frac{2}{3}R_2 + R_3 \rightarrow R_3$

$\frac{5}{3}R_2 + R_4 \rightarrow R_4$

no change

$$\begin{vmatrix} 1 & 0 & -2 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & -13/3 & 17/3 \\ 0 & 0 & 11/3 & -16/3 \end{vmatrix} = 1 \times 3 \begin{vmatrix} -13/3 & 17/3 \\ 11/3 & -16/3 \end{vmatrix} =$$

$$3\left(\frac{208}{9} - \frac{187}{9}\right) = 3\left(\frac{21}{9}\right) = 7$$

(-1) change det of original $= \boxed{-7}$

3. Determine if the following sets are linearly independent or dependent. Justify your answers **without performing matrix calculations.** (3 points each)

a. $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ -4 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -6 \\ -19 \end{bmatrix} \right\}$

not independent
too many vectors
for $\mathbb{R}^4$

b. $\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}} \right\}$

independent
no multiples or
linear combinations

$v_1 \neq -2v_3$

4. Given that A and B are $n \times n$ matrices with det A = -7 and det B = -2, find the following. (3 points each)

a) det (AB) $= (-7)(-2) = 14$

d) det (B^T) $= -2$

b) det (A^{-1}) $= -\frac{1}{7}$

e) det (5A) $= 5^n(-7)$

c) det ($-AB^6$) $= (-1)^n(-7)(-2)^6 =$
$= (-1)^n(-448)$

f) det ($A^{-1}BA$) $= (-\frac{1}{7})(-2)(-7) =$
-2

5. Determine if each statement is True or False. (2 points each)

a. T F If matrix B is formed by multiplying a row of matrix A by -1, then det B = -det A

b. T F The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution when there is at least one free variable.

c. T F If an $m \times n$ matrix has a pivot in every row, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each **b** in R^m.

d. T F If $\{\vec{u}, \vec{v}, \vec{w}, \vec{x}\}$ is linearly independent, then $\vec{u}, \vec{v}, \vec{w}, and\ \vec{x}$ are not in R^3.

e. T F If A and B are $m \times n$ matrices, then both AB^{T} and $A^{\mathrm{T}}B$ are defined.

$(m\times n)\cdot(n\times m)$ $(n\times m)\cdot(m\times n)$

f. T F Interchanging three rows of an $n \times n$ matrix A, you will not change the determinant.

g. T F If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, then so is $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}\}$.

h. T F The pivot columns of a matrix are always linearly dependent.

i. T F If det A is zero, then two rows or two columns of A are the same, or a row or a column is zero.

j. T F If A and B are row equivalent, then their column spaces are the same.

k. T F The vector space $\mathbb{P}_4$ and R^5 are isomorphic.

l. T F A linearly independent set in a subspace H is a basis for H.

m. T F If P_B is the change-of-coordinates matrix, then $\left[\vec{x}\right]_B = P_B^{-1}\vec{x}$ for $\vec{x}$ in V.

n. T F There are only two conditions a vector space must satisfy: it must be closed under addition and closed under multiplication.

o. T F The vector space of 2x3 matrices is isomorphic to $\mathbb{R}^6$.

p. T F The nullspace of an mxn matrix A is a subspace of $\mathbb{R}^m$.

q. T F $(AB)^{-1} = A^{-1}B^{-1}$

r. T F The change of basis matrix is constructed from putting the basis vectors into the rows of P_B.

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Math 2568, Exam #2 – Part 2, Spring 2013

Instructions: On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

1. Determine if the columns of $A=\begin{bmatrix} 1 & 5 & 2 \\ 6 & 4 & -1 \\ -4 & -2 & 1 \\ 3 & 1 & -1 \end{bmatrix}$ form a linearly independent set and justify your answer. (5 points)

there are only 2 pivots in the reduced matrix

rref on A^T = $\begin{bmatrix} 1 & 0 & 2/13 & -3/13 \\ 0 & 1 & -9/13 & 7/13 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So the set is not independent

2. Given $T:\mathrm{R}^4 \to \mathrm{R}^3$ such that $T(\mathbf{x})=\begin{bmatrix} x_1-4x_3+x_4 \\ x_2+2x_3 \\ -x_1+5x_4 \end{bmatrix}$ answer the following.

   a. Find the standard matrix, *A*, such that $T(\mathbf{x})=A\mathbf{x}$. (4 points)

$A=\begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 5 \end{bmatrix}$

b. Is T onto $\mathbb{R}^3$? Justify your answer. (3 points)

reduced matrix has 3 pivots - one in each row - so it is onto

c. Is T one-to-one? Justify your answer. (3 points)

No. There is a free variable

3. Determine if the set $H = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Justify your answer. (5 points)

This matrix (of the basis) reduces to the identity, so yes, it is a basis of $\mathbb{R}^3$

4. Assume that $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 0 \\ 2 & 4 & -5 & 1 & 2 \\ 1 & 2 & 0 & 3 & 1 \\ 3 & 6 & -1 & 8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & -1 & 8 & 1 \\ 0 & 0 & 13 & 13 & -4 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ are row equivalent.

a. Find a basis for the column space of A. (5 points)

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

b. Find a basis for the null space of A. (7 points)

reducing $\begin{bmatrix} 1 & 2/3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\mathrm{Nul}\, A = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$x_1 = -2/3\, x_2 - 3x_4$

$x_2 =$ free $= t$

$x_3 = -x_4$

$x_4 =$ free $= s$

$x_5 = 0$

$\vec{x} = t \begin{bmatrix} -2/3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

c. Determine if $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \end{bmatrix}$ is in Col A. Show appropriate work to justify your answer. (4 points)

augment A (or just vectors of Col A w/ b & reduce)

reduced system is not consistent

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 2 & -5 & 2 & 0 \\ 1 & 0 & 1 & 4 \\ 3 & -1 & 1 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Since $\vec{b}$ is independent of Col A it isn't in Col A

5. Given the basis $\mathcal{B} = \{1-2t^3, t-2t^2, 2-5t+t^2, 3-t^2+7t^3\}$ for P_3. Find $\vec{p}(t) = 6+19t-7t^2$ in this basis. (10 points)

$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -5 & 0 \\ 0 & -2 & 1 & -1 \\ -2 & 0 & 0 & 7 \end{bmatrix} = P_B$

$[\vec{x}]_B = P_B^{-1} \begin{bmatrix} 6 \\ 19 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} 63/113 & 28/113 & 14/113 & -25/113 \\ -10/113 & -17/113 & -65/113 & -5/113 \\ -2/113 & -26/113 & -13/113 & -1/113 \\ 18/113 & 8/113 & 4/113 & 9/113 \end{bmatrix} \begin{bmatrix} 6 \\ 19 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} 812/113 \\ 72/113 \\ -415/113 \\ 232/113 \end{bmatrix}$

this does check out.

6. Given that $\det A^{-1} = \dfrac{1}{\det A}$ if A is invertible, use this fact and the fact that AB is invertible to prove that both A and B must be invertible. [Hint: use multiplication properties of the determinant and what you know about nxn identity matrices.] (10 points)

if AB is invertible then $(AB)^{-1}(AB) = I \Rightarrow B^{-1}A^{-1}AB = I$

taking det. of both sides we get

$\det(B^{-1}A^{-1}AB) = \det(I) = 1$

by the product property of determinants

$\Rightarrow (\det B^{-1})(\det A^{-1})(\det A)(\det B) = 1$

Since 1 is non-zero, we know that none of $\det(A^{-1})$, $\det(B^{-1})$, $\det(A)$ or $\det B$ can be $= 0$.

Since their dets don't $= 0$, they must be invertible.

7. Prove that the following are vector spaces or show that they are not. (5 points each)

a. $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a = b + c; a, b, c \text{ real} \right\}$ is a vector space.

$H = \left\{ \begin{bmatrix} b+c \\ b \\ c \end{bmatrix} \right\}$ a) if $b, c = 0$ then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the set

b) $\begin{bmatrix} b+c \\ b \\ c \end{bmatrix} + \begin{bmatrix} a+d \\ a \\ d \end{bmatrix} = \begin{bmatrix} b+c+a+d \\ b+a \\ c+d \end{bmatrix}$

it's true that $b+c+a+d = (b+a)+(c+d)$ ✓

c) $k \begin{bmatrix} b+c \\ b \\ c \end{bmatrix} = \begin{bmatrix} k(b+c) \\ kb \\ kc \end{bmatrix}$ it's true that $k(b+c) = kb + kc$ ✓

b. $W = \left\{ \begin{bmatrix} a & a+2 \\ b & c \end{bmatrix}, a, b, c \text{ real} \right\}$

this is not a vector space

Since if, $a, b, c = 0 \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ so $\vec{0}$ not in the set.